Warsaw University of Technology

Faculty of Mathematics and Information Science

Doctor of Philosophy Dissertation

Selected reproducing kernels: admissible weights and dependence on parameters

by

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Ad maiorem Dei gloriam, Warsaw, 2022.

To prof. Feliks Koneczny, outstanding scholar, creator of the original system of the comparative science of civilizations and Tadeusz Kościuszko, the purest son of liberty

Acknowledgments

I am dearly obliged to people with whom I had inspiring discussions in the topic of the dissertation, those professors which took care of me and people which helped me in any other way.

Synopsis

At the beginning elements of the theory of reproducing kernel Hilbert spaces will be recalled. It will be shown that existence of the reproducing kernel is equivalent to continuity of the functionals of point evaluation. The proof that if $\mathcal{H}$ is a reproducing kernel Hilbert space of functions defined on U, then in any set

$$\{f \in \mathcal{H} \mid f(z) = 1\}, z \in U$$

if non-empty, there is exactly one element with minimal norm, will be given.

The second chapter will be devoted to the Hilbert spaces of square-integrable functions which are the kernel (in the algebraic sense) of some elliptic operator. It will be shown that such a space is a reproducing kernel Hilbert space. Then we will prove that the reproducing kernel of that space depends in a continuous way on a weight of integration, i.e. on a deformation of an inner product. Convergence of weights only almost everywhere will be needed. Next we will generalize Ramadanov theorem, i.e. we will show that the reproducing kernel of such a space depends in continuous way on a domain of integration, i.e. on a domain on which our functions are defined. It will be done in three different ways for the case an increasing sequence of domains. Moreover sufficient condition for the case of decreasing sequence of domains will be given.

Particular case of such a Hilbert space is Hilbert space of square-integrable and harmonic functions. In such a case it will be shown that if only an inverse of a weight of integration is integrable in some positive power, then the reproducing kernel of the corresponding weighted Hilbert space exists. Moreover an example of a weight for which there is no reproducing kernel of such a space will be given.

By the minimal norm property of the reproducing kernel recalled in the first chapter, we will conclude that in the set of square-integrable solutions of an elliptic equation, which take value at some given point equal to c, if non-empty, there is exactly one element with minimal norm. Moreover such an element depends in continuous way on a weight and domain of integration in a precisely defined sense.

The third chapter will be devoted to the weighted kernels of Szegö type. We will give sufficient conditions for a weight of integration in order for the reproducing kernel of the weighted Szegö space to exist. In particular it will be shown that if an inverse of a weight of integration is integrable, then there exists the reproducing kernel of the corresponding weighted Szegö space. The case of domains with non-connected boundaries will be also considered. Moreover we will give an example of a weight for the unit ball for which there is no reproducing kernel of the corresponding space. Using biholomorphisms we will prove that such weights exist for a large class of domains.

Then we will prove that Szegö kernel depends in a continuous way on a weight of integration. Pasternak's theorem on dependence of the orthogonal projector on a deformation of an inner product will be used in the proof. Finally it will be shown how weighted Szegö kernel can be used to prove general theorems of complex analysis.

Keywords: Reproducing kernel Hilbert space, functional of point evaluation, reproducing kernel, elliptic operator, elliptic equation, minimal solution, Szegö kernel, admissible weights, weights of integration, dependence on parameters, continuous dependence, Ramadanov theorem, continuous dependence on a weight of integration.

Streszczenie

Na początku przywołane zostaną elementy teorii przestrzeni Hilberta z jądrem reprodukującym. Zostanie pokazane, że istnienie jądra reprodukującego jest równoważne ciągłości funkcjonałów ewaluacji. Dany będzie dowód, że jeśli $\mathcal{H}$ jest przestrzenią Hilberta z jądrem reprodukującym funkcji określonych na U, wówczas w dowolnym zbiorze

$$\{f \in \mathcal{H} \mid f(z) = 1\}, z \in U$$

o ile jest niepusty, znajduje się dokładnie jeden element o minimalnej normie.

Drugi rozdział poświęcony zostanie przestrzeniom Hilberta funkcji całkowalnych z kwadratem, które są jądrem (w sensie algebraicznym) pewnego operatora eliptycznego. Będzie pokazane, że taka przestrzeń jest przestrzenią Hilberta z jądrem reprodukującym. Potem pokażemy, ze jądro reprodukujące takiej przestrzeni zależy w sposób ciągły od wagi całkowania, tzn. od deformacji iloczynu skalarnego. Zbieżność wag zaledwie prawie wszędzie będzie potrzebna. Następnie uogólnimy twierdzenie Ramadanowa, tzn. pokażemy, że jądro reprodukujące takiej przestrzeni zależy w sposób ciągły od obszaru całkowania, tzn. od dziedziny, na której określone są nasze funkcje. Zostanie to zrobione na trzy sposoby dla przypadku rosnącego ciągu obszarów. Ponadto zostanie podany warunek wystarczający dla przypadku malejącego ciągu obszarów.

Szczególnym przypadkiem takiej przestrzeni jest przestrzeń funkcji całkowalnych z kwadratem i harmonicznych. Dla tego przypadku będzie pokazane, że jeśli tylko waga całkowania jest całkowalna w jakiejś dodatniej potędze, wówczas jądro reprodukujące

odpowiadającej ważonej przestrzeni Hilberta istnieje. Co więcej podany będzie przykład wagi, dla której nie istnieje jądro reprodukujące takiej przestrzeni.

Korzystając z własności minimalnej normy jądra reprodukującego przywołanej w rozdziale drugim, stwierdzimy, że w zbiorze całkowalnych z kwadratem rozwiązań równania eliptycznego, które w pewnym zadanym punkcie przyjmują wartość c, o ile jest niepusty, znajduje się dokładnie jeden element o minimalnej normie. Co więcej, ten element zależy w sposób ciągły od wagi i od obszaru całkowania w ściśle określonym sensie.

Trzeci rozdział poświęcony zostanie ważonym jądrom typu Szegö. Damy warunki wystarczające na wagę całkowania, aby istniało jądro reprodukujące odpowiadającej ważonej przestrzeni. W szczególności będzie pokazane, że jeśli odwrotność wagi całkowania jest całkowalna, to istnieje jądro reprodukujące odpowiadającej ważonej przestrzeni Szegö. Przypadek obszarów o niespójnych brzegach również będzie rozważony. Co więcej, damy przykład wagi dla kuli jednostkowej, dla której nie istnieje jądro reprodukujące odpowiadającej przestrzeni. Używając bihomorfizmów, pokażemy, że takie wagi istnieją dla szerokich klas obszarów.

Później będzie udowodnione, że jądro Szegö zależy w sposób ciągły od wagi całkowania. Twierdzenie Pasternaka o zależności rzutu ortogonalnego od deformacji iloczynu skalarnego zostanie użyte w dowodzie. Na koniec zostanie pokazane, jak ważone jądro Szegö może być użyte do udowodnienia ogólnych twierdzeń analizy zespolonej.

Słowa kluczowe: Przestrzeń Hilberta z jądrem reprodukującym, funkcjonał ewaluacji, jądro reprodukujące, operator eliptyczny, równanie eliptyczne, minimalne rozwiązanie, jądro Szegö, wagi dopuszczalne, wagi całkowania, zależność od parametrów, ciągła zależność, twierdzenie Ramadanowa, ciągła zależność od wagi całkowania.

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Introduction

First papers in the theory of reproducing kernels were published one hundred years ago by S. Zaremba, J. Mercer, S. Bochner, G. Szegö and S. Bergman (see [Zaremba1907], [Mercer1909], [Bochner 1918], [Szegö 1921], [Bergman1922]; see also [Szafraniec2016], in polish, for more details.) Now we know, however, that concept of a reproducing kernel is more general and can be associated with an arbitrary Hilbert space of functions. Milestone in the theory of reproducing kernels was paper by N. Aronszajn (see [Aronszajn 1950]). For more modern approach see [Paulsen2016] or [Szafraniec 2004], the second one is in polish.

In the first chapter for the reader's convenience we recall elements of the theory of reproducing kernel Hilbert spaces which will be used later. We show that the existence of the reproducing kernel of a Hilbert space of functions is equivalent to the continuity of functionals of point evaluation in that space. We also prove the 'minimal norm property', i.e. the fact that if the set of functions from our reproducing kernel Hilbert space which have the value at some point equal to 1 is not empty, then there is exactly one function with minimal norm.

The second chapter is devoted to reproducing kernel Hilbert spaces of solutions of uniformly elliptic equation. V. M. Malyshev in 1997 (see [Malyshev1997]) considered continuous embeddings of square-integrable functions from a kernel of hypoelliptic operator into the space of continuous functions. In this dissertation we will also consider Hilbert spaces connected with the kernel of elliptic operator, but our space and reproducing kernel will be a different idea.

It can be shown that if an operator in the consideration is elliptic, then there exists the reproducing kernel of Hilbert space of square-intregrable functions which are elements of the kernel of some elliptic operator. Theory of Sobolev spaces is used there. We prove that the dependence of such a kernel on a weight of integration, i.e. on a deformation of an inner product, is continuous. Note that Z. Pasternak-Winiarski showed that a classical Bergman kernel depends even in an analytic way on a weight of integration (see [Pasternak1990]), but he assumed that weights converge in a pretty strong topology, while we need only convergence of weights almost everywhere in the proof.

Next we show that a reproducing kernel of Malyshev type depends in the continuous way on an increasing sequence of domains of integration, generalizing well-known Ramadanov Theorem (see [Ramadanov1967]). It is done in three different ways. One of them uses the 'minimal norm property'. The second one uses the idea of connection between a reproducing kernel and an orthogonal projector — it is a generalization of Skwarczyński's results (see [Skwarczyński1985a], [Skwarczyński1985b]). The third method is a new idea which uses orthogonal projectors and weak convergence.

We also give sufficient conditions for Ramadanov Theorem for a decreasing sequence of domains to be true.

Bearing in mind the minimal norm property of a reproducing kernel, we conclude that in the set of square-integrable solutions of elliptic equations which have value equal to some c in a given point, if not empty, there is exactly one element with minimal L^2-norm. Moreover this element depends in continuous way on a weight of integration and a domain of integration in a precisely defined sense. It looks like this theorem was not known before; note that usually name 'minimal solution of a differential equation' is used to denote different kinds of extremal solutions.

Particular case of our space is the Hilbert space of square-integrable harmonic functions. Much is known about the reproducing kernel of such a space, including direct formula when the domain is the unit ball (see e.g. [Ramey1996], [Axler2001], [Kang2001], [Koo2005]). For that case we generalize Z. Pasternak-Winiarski's results on admissible weights (see [Pasternak1992]). We show which conditions a weight of integration must satisfy for the reproducing kernel of a corresponding weighted space to exist, in particular we show that if an inverse of a weight is integrable in some positive power, then the reproducing kernel exists. We also give an example of a weight for which a reproducing kernel of such a space does not exist. Moreover we show an upper estimate for a minimal solution of Laplace's equation in the sense described above.

The next part of the dissertation is devoted to the case of the Szegö kernel. Such a reproducing kernel plays an important role in mathematics. For example, it is known that for simply connected domains in $\mathbb{C}^1$ there is a direct connection between Poisson kernel and Szegö kernel (see e.g. [Stein1972] and the last section of the dissertation). Moreover the unique function from Riemann Mapping Theorem may be given using the Szegö kernel (see [Bell2015]).

In what follows we will consider weighted Szegö kernels. Properties of such a generalization of classical Szegö kernel were investigated in few papers (see e.g. [Nehari1952], [Alenitsin1972], [Uehara1984], [Uehara1995]; the second paper is in russian). In all of them, however, only continuous weights were in the consideration. Therefore it is natural to prove some Theorems which state how Szegö kernel depends on a weight of integration in the case when weights do not have to satisfy this assumption. Before doing that, we need to answer the question which weights are 'good enough' to take, i.e. for which weights there exists a reproducing kernel of corresponding weighted Szegö space. The case of Szegö space is more subtle than the case of classical and harmonic Bergman spaces, as we will see in this section.

We find sufficient conditions for a weight in order for the Szegö kernel of the corresponding weighted Szegö space to exist. We show that if an inverse of a weight is integrable, then the Szegö kernel exists. A case of weights for domains with non-connected boundaries is also considered. We give an example of a weight on the unit ball, for which the Szegö kernel of the corresponding Szegö space does not exist. Moreover using biholomorphisms we show that such weights exist for a large class of domains. Bell-Ligocka Theorem on existence of a smooth prolongation of a biholomorphism to the boundary (see [Bell1980]) is used in this proof. This section is mainly based on [Żynda2020].

Then we show that the Szegö kernel depends in continuous way on a weight of integration, i.e. on a deformation of an inner product. Pasternak's Theorem on a dependence of orthogonal projection on a deformation of an inner product (see [Pasternak1998]) is used in the proof.

We also show how a weighted Szegö kernel can be used to prove elementary theorems of complex analysis. This part is based on [Żynda2019b].

Chapter 1

Elements of the theory of reproducing kernels

The aim of this chapter is to recall basic concepts of the theory of reproducing kernel Hilbert spaces which will be used later.

1.1 Reproducing kernel Hilbert space

Definition 1.1. *Let $\mathcal{H}$ be a Hilbert space of complex-valued functions defined on the same domain U with an inner product $\langle\cdot|\cdot\rangle$ with complex conjugate on the first variable and let $||\cdot||$ be a norm induced by that inner product. The function $K : U \times U \to \mathbb{C}$, if it exists, such that for any $z \in U$ and for any $f \in \mathcal{H}$ we have:*

(i) $\overline{K(z,\cdot)} \in \mathcal{H}$;

*(ii) (**reproducing property**) $\langle\overline{K(z,\cdot)}|f\rangle = f(z)$;*

*will be called **a reproducing kernel** of a Hilbert space $\mathcal{H}$.*

Note that of course each real-valued function is in fact complex-valued function.

It is easy to see that in particular Hilbert spaces $\mathbb{R}^N$ and $\mathbb{C}^N$ with inner products defined in the classical way:

$$\langle x|y\rangle := \sum_{i=1}^{N} \overline{x_i} y_i$$

for $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N)$ can be treated as Hilbert spaces of functions. Indeed, we can treat these spaces as spaces of functions defined on $\{1, 2, \dots, N\}$ with values in $\mathbb{R}$ and $\mathbb{C}$, respectively. Such a Hilbert space is equipped with the reproducing kernel

$$K(z, w) = \chi_z(w),$$

where $\chi_z(w)$ is equal to 1 when $w = z$ and 0 otherwise.

We can also consider the space $l^2(\mathbb{C}, \mu)$ of complex sequences $(x_1, x_2, \dots)$, square-summable in the sense:

$$||x||_\mu := \sum_{i=1}^{+\infty} |x_i|^2 \mu_i < \infty$$

for some $\mu = (\mu_1, \mu_2, \dots)$, such that $\mu_i > 0$ for any $i \in \mathbb{N}$. Such a space with an inner product:

$$\langle x_i|y_i\rangle_\mu = \sum_{i=1}^{+\infty} \overline{x_i} y_i \mu_i$$

is a Hilbert space equipped with the reproducing kernel:

$$K(z, w) = \frac{1}{\mu_z} \chi_z(w),$$

where χ_z was defined in the example above. (See [Żynda2019a] for more details.)

Although the cases of $\mathbb{C}^n$ and $l^2(\mathbb{C}, \mu)$ are easy to investigate, general formulas for any reproducing kernel are not known. Therefore it is important to prove theorems which show how reproducing kernels depend on different parameters, which we do in this dissertation for Malyshev and Szegö kernels.

It is well-known that for any reproducing kernel K we have

$$K(z, z) \geq 0.$$

for $z \in U$. Indeed, by the reproducing property and the fact that the norm of any function is non-negative we have:

$$K(z,z) = \langle \overline{K(z,\cdot)} | \overline{K(z,\cdot)} \rangle = ||\overline{K(z,\cdot)}||^2 \geq 0.$$

This fact will be used frequently in what follows without further reminding.

We can also prove something stronger:

Proposition 1.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions defined on a domain U, such that the function $\mathcal{I}$ equal to 1 everywhere is its element. Then for any $z \in U$ we have*

$$K(z,z) \geq \frac{1}{||\mathcal{I}||^2}.$$

Proof: By the reproducing property and Cauchy's inequality:

$$1 = |\langle \overline{K(z,\cdot)} | \mathcal{I} \rangle| \leq ||\overline{K(z,\cdot)}|| \cdot ||\mathcal{I}|| = ||\mathcal{I}|| \sqrt{K(z,z)}. \blacksquare$$

Proposition 1.2. *For any reproducing kernel K it is true that*

$$|K(z,w)| \leq \sqrt{K(z,z)} \sqrt{K(w,w)}.$$

Proof: By the reproducing property and Cauchy's inequality:

$$|K(z,w)| = |\langle \overline{K(w,\cdot)} | \overline{K(z,\cdot)} \rangle| \leq ||\overline{K(z,\cdot)}|| \cdot ||\overline{K(w,\cdot)}||$$

Using the reproducing property again we get the thesis. $\blacksquare$

Note that not each Hilbert space of functions is equipped with a reproducing kernel. Examples of Hilbert spaces of functions without corresponding reproducing kernels can be found in [Pasternak1992] or [Żynda2020]. We will show results of the second paper later in this dissertation. We will also give an example of Malyshev space which is not a reproducing kernel Hilbert space.

Theorem 1.1. *The following conditions are equivalent:*

(i) there exists a reproducing kernel of $\mathcal{H}$*;*

(ii) functionals of point evaluation

$$\mathcal{H} \ni f \mapsto f(z) \in \mathbb{C}$$

are continuous for any $z \in U$*, i.e. for any* $z \in U$ *there exist* $C_z > 0$*, such that for any* $f \in \mathcal{H}$

$$|f(z)| \leq C_z ||f||. \tag{1.1}$$

Proof: (i) $\Rightarrow$ (ii) By the reproducing property and Cauchy-Schwarz inequality

$$|f(z)| = |\langle \overline{K(z,\cdot)} | f \rangle| \leq ||\overline{K(z,\cdot)}|| \cdot ||f|| = \sqrt{K(z,z)} ||f||.$$

(ii) $\Rightarrow$ (i) If functionals of point evaluation are continuous, then by Riesz representation theorem for each $z \in U$ there exists $\overline{e_z} \in \mathcal{H}$, such that

$$f(z) = \langle \overline{e_z} | f \rangle.$$

Function K defined in the following way

$$K(z,w) := e_z(w)$$

is the reproducing kernel of the Hilbert space $\mathcal{H}$. ■

Another consequence of the Riesz representation theorem is the fact that if the reproducing kernel of a Hilbert space exists, then it is unique.

It is well-known that each two Hilbert spaces with complete orthonormal systems of the same cardinality are isometrically isomorphic. However, it is possible that in one of them functionals of point evaluation are continuous, while in another — not, i.e. it is possible that out of two isometrically isomorphic Hilbert spaces one is a reproducing kernel Hilbert space, while the other one — is not.

Note that each finite-dimensional Hilbert space is a reproducing kernel Hilbert space. Indeed, since any linear operator between two finite-dimensional Banach spaces is continuous, so in particular it applies to any functional of point evaluation.

Proposition 1.3. *$\sqrt{K(z,z)}$ is the smallest constant C_z, for which inequality (1.1) holds.*

Proof: Let $E_z : \mathcal{H} \ni f \mapsto f(z) \in \mathbb{C}$ be the functional of point evaluation. By the Riesz correspondence theorem,

$$||E_z||^* = ||\overline{K(z,\cdot)}||,$$

but

$$||\overline{K(z,\cdot)}|| = \sqrt{K(z,z)}.$$

At once $||E_z||^*$ is by definition the smallest constant for which inequality (1.1) holds. ■

As we will see in what follows, the case of kernel of Szegö type is more subtle and we will need to change general theory introduced above a bit to suit it to that special case.

1.2 Minimal norm property

In this whole section we assume that $\mathcal{H}$ is a Hilbert space of functions defined on U and K is its reproducing kernel.

Theorem 1.2. *The following conditons are equivalent for a point $z \in U$:*

(i) $f(z) = 0$ for any $f \in \mathcal{H}$;

(ii) $K(z,z) = 0$;

(iii) $K(z,\cdot) \equiv 0$.

Proof: (i) $\Rightarrow$ (ii) If for some $z \in U$ we have $f(z) = 0$ for any $f \in \mathcal{H}$, then in particular for $g(\cdot) = \overline{K(z,\cdot)}$ we have $g(z) = 0$.

(ii) $\Rightarrow$ (iii) Because

$$\langle \overline{K(z,\cdot)} | K(z,\cdot) \rangle = K(z,z) = 0$$

and the only element of any Hilbert space with a norm equal to zero is zero, we have $K(z,\cdot)\equiv 0$ on U.

(iii)$\Rightarrow$ (i) By the reproducing property, for any $f\in\mathcal{H}$ we have

$$f(z)=\langle\overline{K(z,\cdot)}|f\rangle=0. \blacksquare$$

Theorem 1.3. *Let K be a reproducing kernel of $\mathcal{H}$. If $K(z,z)\neq 0$, then*

$$k_z(\cdot):=\frac{\overline{K(z,\cdot)}}{K(z,z)}$$

is the only element of $\mathcal{H}$ with the following properties:

(i) $k_z(z)=1$;

(ii) if $m_z\in\mathcal{H}$, $m_z(z)=1$ and $||m_z||\leq||k_z||$, then $m_z=k_z$. Moreover

$$||k_z||=\frac{1}{\sqrt{K(z,z)}}.$$

Proof: By Theorem 1.2 there exists $f\in\mathcal{H}$, such that $f(z)\neq 0$. By inequality (1.1) and Proposition 1.3 we have for such a function f

$$\frac{1}{\sqrt{K(z,z)}}\leq\frac{||f||}{|f(z)|}=\left|\left|\frac{f}{f(z)}\right|\right|. \tag{1.2}$$

But

$$\left|\left|\frac{\overline{K(z,\cdot)}}{K(z,z)}\right|\right|^2=\frac{1}{K(z,z)}$$

by the reproducing property. To end the proof we need only to show that, if $||m_z||=||k_z||$, then $m_z=k_z$. Note that for $f_z:=\frac{1}{2}(m_z+k_z)$ we have $f_z(z)=1$ and

$$||f_z||=\left|\left|\frac{1}{2}(m_z+k_z)\right|\right|\leq\frac{1}{2}(||m_z||+||k_z||)=||k_z||.$$

On the other hand we showed above that

$$||f_z||\geq||k_z||,$$

(see (1.2)), so $||f_z||=||k_z||$. Since in our case the triangle inequality is in fact an equality and each Hilbert space is strictly convex, there exists $\alpha\in\mathbb{C}$, such that $m_z=\alpha k_z$. Thus

$$||f_z||=\left|\left|\frac{1}{2}(m_z+k_z)\right|\right|=\frac{1}{2}(\alpha+1)||k_z||.$$

Since

$$||f_z|| = ||k_z||,$$

we see that $\alpha = 1$ and in conclusion $m_z = k_z$. ■

Chapter 2

Kernels of Malyshev type

V. A. Malyshev considered (see [Malyshev1997]) continuous embeddings of square-integrable functions from a kernel of hypoelliptic operator into the space of continuous functions. In this dissertation we will also consider Hilbert spaces connected with the kernel of elliptic operator, but our space and reproducing kernel will be a different idea.

Recalling minimal norm property of reproducing kernels from chapter 1.2., we find out that reproducing kernel of such a type is a powerful tool which allows us to solve extremal problems for solutions of elliptic equations.

In whole this chapter by 'derivative' we understand 'weak derivative' i.e. in the sense of distribution theory and all solutions are understood in the strong sense.

2.1 Elements of Partial Differential Equations Theory

The aim of this section is to recall some classical results of Partial Differential Equations Theory, which will be used later.

Definition 2.1. *Name* ***multiindex of length*** $n \in \mathbb{N}$ *will be used to denote any element* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0, 1, 2, \dots\}^n$. *Its* ***absolute value*** *will be an expression*

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

Let α be a multiindex of length n. For convenience we will use the following notation:

$$f^{(\alpha)} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Definition 2.2. *If for a given function $f : \mathbb{R}^n \supset U \to \mathbb{R}$ we have*

$$Df = -\sum_{i,j=1}^{n} (a^{ij}(x)(f_{x_i}))_{x_j} + \sum_{i=1}^{n} b^i(x) f_{x_i} + c(x)f,$$

*and $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, then we will say that D is **a differential operator of order** 2 **in its divergence form**.*

Definition 2.3. *If there exists a constant $\Theta > 0$, such that*

$$\sum_{i,j=1}^{n} a^{ij}(x) v_i v_j \geq \Theta |v|^2,$$

*for any $x \in U$ and any $v \in \mathbb{R}^n$ in the definition above, then we will say that D is **an elliptic operator**.*

Often such a name is used to denote uniformly elliptic operator. See e.g. [Evans1998] for more details.

Ellipticity means that matrix

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) & \ldots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \ldots & a_{2n}(x) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(x) & a_{n2}(x) & \ldots & a_{nn}(x) \end{pmatrix}$$

is positively defined for any $x \in U$. Note that ellipticity depends only on coefficients of second-order partial derivatives.

Example: The Laplace's operator in $\mathbb{R}^2$

$$\triangle := -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$$

is a classical example of an elliptic operator. Indeed, inequality from the definition of an elliptic operator is satisfied for $\Theta = 1$ and in fact becomes equality.

Definition 2.4. *Let $k \in \mathbb{N}$, $1 \le p \le \infty$ and U be a domain in $\mathbb{R}^n$. The Sobolev space $W^{k,p}(U)$ is defined as the set of all functions f defined on U, such that for every multi-index α with $|\alpha| \le k$, the mixed partial derivative $f^{(\alpha)}$ exists in the weak sense and is an element of $L^p(U)$.*

Proposition 2.1. *$W^{k,p}(U)$ with a norm*

$$[[f]]_U^{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} \left\|f^{(\alpha)}\right\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} & 1 \leqslant p < \infty; \\ \max_{|\alpha|\leqslant k} \left\|f^{(\alpha)}\right\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

is a Banach space.

We omit the proof.

Definition 2.5. *Let $U \subset \mathbb{R}^n$ be a domain. The Hölder's space $C^{k,\gamma}(U)$ for $k \in \mathbb{N}$, $\gamma \in (0,1]$, is defined as a set of functions f from $C^k(U)$ for which Hölder's norm*

$$]]f[[_U^{k,\gamma} := \sum_{|\alpha|\le k} \sup_{x\in U} |f^{(\alpha)}| + \sum_{|\alpha|=k} \sup_{x,y\in U; x\neq y} \left\{ \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x-y|^\gamma} \right\}$$

is finite.

Proposition 2.2. *The Hölder's space $C^{k,\gamma}(U)$ is a Banach space.*

As in the case of Sobolev spaces, we omit the proof. For more extensive treatment of the subject see e.g. [Adams2003].

Theorem 2.1. *Let $U \subset \mathbb{R}^n$ be a domain with the boundary of class C^1. Let f be an element of the Sobolev space $W^{k,p}(U)$. If $k > \frac{n}{p}$, then f is also an element of the Hölder's space*

$$C^{k-[n/p]-1,\gamma}(\overline{U}),$$

where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{Z}; \\ \text{any positive number less than 1}, & \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

Moreover there exists a constant $C_1 > 0$, such that

$$]]f[[_U^{k-\left[\frac{n}{p}\right]-1,\gamma} \leq C_1[[f]]_U^{k,p}.$$

For more details, see [Evans1998], Theorem 6 in Section 5.6.3.

Theorem 2.2. *Let U be a domain in $\mathbb{R}^n$ with a boundary of class C^1. Let D be an elliptic operator such that (**in divergence form**)*

$$Du = -\sum_{i,j=1}^{n} (a^{ij}(u_{x_i}))_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i}, +c(x)u$$

where $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$. Let $f \in L^2(U)$ and $u \in W^{1,2}(U)$ be a weak solution of elliptic equation

$$Du = f$$

in U. Then $u \in W^{2,2}(V)$ for any compact set $V \subset U$ and there exists constant $C_2 > 0$, such that

$$[[u]]_V^{k,p} \leq C_2\left(||f||_U + ||u||_U\right).$$

For more details, see [Evans1998], Theorem 1 in Section 6.3.1.

Moreover

Theorem 2.3. *Let $a^{ij}, b^i, c \in C^\infty(U)$ and $f \in C^\infty(U)$. Let $u \in W^{1,2}(U)$ be a weak solution of elliptic equation*

$$Du = f$$

in U. Then $u \in C^\infty(U)$, which means that it is in fact strong solution.

For more details see [Evans1998], Theorem 3 in Section 6.3.

2.2 Space and kernel of Malyshev type

Let U be a domain in $\mathbb{R}^N$. Let $L^2(U)$ denote the space of classes of measurable functions defined on U such that

$$||f||_U^2 := \int_U |f(w)|^2 \mathrm{d}w < \infty.$$

Such a space with an inner product

$$\langle f|g\rangle_U := \int_U \overline{f(w)}g(w)\mathrm{d}w$$

is a Hilbert space.

(If U does not change in our considerations or is unspecified, we will simplify our notation to $||\cdot||$ and $\langle -,\cdot\rangle$.)

Let now D be a linear differential operator defined on $L^2(U)$. By $L^2D(U)$ we understand

$$\{f \in L^2(U) : Df = 0\},$$

where the equality is understood in the strong sense.

Note that if coefficients of operator D are of class C^∞, then by Theorem 2.3 weak solution of the equation $Df = 0$ is in fact strong solution, which means that we can identify elements of $L^2D(U)$ with their continuous representants, so in particular value of these functions in any point $z \in U$ will be well defined.

Proposition 2.3. *Let f_n be a sequence of functions such that $Df_n = 0$ for any n convergent to function f in $L^2(U)$ topology. Then $Df = 0$. Moreover if coefficients of operator D are of class C^∞, then the space $L^2D(U)$ is a closed subspace of $L^2(U)$.*

Proof: Let $f_n \in L^2D(U)$ and suppose that $f_n \to f$ in the $L^2(U)$ topology. Let h be an element of some dense subspace of $L^2(U)$ contained in the domain of D^*. Then

$$0 = \langle h|Df_n\rangle = \langle D^*h|f_n\rangle,$$

and

$$0 = \langle D^*h|f\rangle = \langle h|Df\rangle,$$

which implies

$$0 = \langle h|Df\rangle.$$

Since h was chosen arbitrarily from a dense subspace of $L^2(U)$, $Df = 0$. Moreover if D has smooth coefficients, then f is also the strong solution and $L^2D(U)$ is closed in $L^2(U)$. ■

So we know that if only D has coefficients of C^∞ class, then $L^2D(U)$ is a Hilbert space. Such a Hilbert space is a generalization of the well-known Bergman space. Indeed, for example for $U \subset \mathbb{R}^2$ and

$$D = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

$L^2D(U)$ is a space of holomorphic and square-integrable functions on U.

Theorem 2.4. *Let U be a domain in $\mathbb{R}^2$ with the boundary of class C^1. Let D be an elliptic operator, such that (**in divergence form**)*

$$Df = -\sum_{i,j=1}^{2}(a^{ij}(f_{x_i}))_{x_j} + \sum_{i=1}^{2} b^i(x) f_{x_i} + c(x)f,$$

where $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$. Then there exists a reproducing kernel of $L^2D(U)$.

If coefficients of operator D are not of class C^∞, then it is possible that the space of square integrable solutions in the strong sense of the equation $Df = 0$ is not closed. In such a case we can take the closure and define the reproducing kernel on it, using standard techniques (see [Szafraniec2004]).

From now on K_U will be used to denote the reproducing kernel of $L^2D(U)$, especially when the domain U changes in our considerations. Moreover in the remainder of this dissertation, if we say "elliptic operator", we will mean that $U \subset \mathbb{R}^2$ is a domain with C^1-boundary and coefficients in its divergence form that satisfy the hypotheses of the theorem above. We will also write "elliptic equation $Df = 0$" in the same manner.

Proof of the Theorem: Let $f \in L^2D(U)$. Since we consider strong solutions only, f is also an element of $W^{2,2}(V)$ for any compact set $V \subset U$. Let $w \in \text{int}V$. Let r be sufficiently small for the ball $B(w, r) = \{z \in \mathbb{R}^2 : |w - z| < r\}$ to lie in V. Then, for any

$z \in B(w,r)$ by Theorem 2.1, we have

$$|f(z)| \leq C_{B(w,r)}[[f]]^{2,2}_{B(w,r)},$$

where $C_{B(w,r)}$ does not depend on $f \in L^2D(U)$.

By Theorem 2.2,

$$[[f]]^{2,2}_{B(w,r)} \leq C_2||f||_V.$$

Of course

$$||f||_V \leq ||f||_U.$$

So we have shown that for any compact set $X \subset U$ there exists C_X, such that for any function $f \in L^2D(U)$ and for any $w \in X$

$$|f(z)| \leq C_X||f||_U. \tag{2.1}$$

This means that functionals of point evaluation, i.e., functionals

$$E_z : L^2D(U) \ni f \mapsto f(z) \in \mathbb{C}$$

are continuous. Using Theorem 1.1 completes the proof. ■

As we said in chapter 1.1., not every Hilbert space of functions is a reproducing kernel Hilbert space. So let us give an example of a Malyshev space which is not equipped with a reproducing kernel.

Example: Let U be the unit circle in $\mathbb{R}^2$ and

$$D = \frac{\partial^2}{\partial x \partial y}.$$

(Clearly D is not an elliptic operator. To find out that we just need to take e.g. a vector $v = (1, -1)$ to see that the inequality from the definition of an elliptic operator is not satisfied.) It is easy to see that $Df = 0$ in the strong sense if and only if

$$f(x,y) = c(x) + d(y)$$

for some C^2-functions c, d. In particular

$$h_n(x,y) := \exp(-x^2 n) + \exp(-y^2 n) \in L^2 D(U)$$

for any $n \in \mathbb{N}$. We have

$$\begin{aligned}\int_{-1}^{1} \exp(-x^2 n)\mathrm{d}x = \wr x\sqrt{n} = t \wr = \frac{1}{\sqrt{n}} \int_{-\sqrt{n}}^{\sqrt{n}} \exp(-t^2)\mathrm{d}t \\ < \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \exp(-t^2)\mathrm{d}t \to 0.\end{aligned}$$

Similarly

$$\int_{-1}^{1} \exp(-2x^2 n)\mathrm{d}x \to 0.$$

Let

$$S = \{z = (x,y) \in \mathbb{R}^2 : -1 < x < 1,\ -1 < y < 1\}.$$

Then

$$\begin{aligned}
&\int_U |\exp(-x^2 n) + \exp(-y^2 n)|^2 \mathrm{d}x\mathrm{d}y \\
&= \int_U (\exp(-2x^2 n) + \exp(-2y^2 n) + 2\exp(-x^2 n)\exp(-y^2 n))\mathrm{d}x\mathrm{d}y \\
&< \int_S (\exp(-2x^2 n) + \exp(-2y^2 n) + 2\exp(-x^2 n)\exp(-y^2 n))\mathrm{d}x\mathrm{d}y \\
&= \int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} \exp(-2x^2 n)\mathrm{d}x + \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \exp(-2y^2 n)\mathrm{d}y \\
&\quad + \int_{-1}^{1} \exp(-x^2 n)\mathrm{d}x \int_{-1}^{1} \exp(-y^2 n)\mathrm{d}y \\
&= 2\int_{-1}^{1} \exp(-2x^2 n)\mathrm{d}x + 2\int_{-1}^{1} \exp(-2y^2 n)\mathrm{d}y + 2\Big(\int_{-1}^{1} \exp(-x^2 n)\mathrm{d}x\Big)^2 \\
&= 4\int_{-1}^{1} \exp(-x^2 n)\mathrm{d}x + 2\Big(\int_{-1}^{1} \exp(-x^2 n)\mathrm{d}x\Big)^2 \to 0.
\end{aligned}$$

We showed that $||h_n||_U \to 0$. On the other hand $h_n(0,0) = 2$ for any $n \in \mathbb{N}$. It means that the functional of point evaluation for $z = (0,0)$ is not continuous and therefore inequality (1.1) is not true for that point. By Theorem 1.1, $L^2 D(U)$ is not equipped with a reproducing kernel. ■

Note that the author of [Malyshev1997] considered different kind of space and reproducing kernel connected with the kernel of an elliptic operator. He claimed that the space of square-integrable functions which are elements of the kernel of hypoelliptic operator can be continuously embedded in the space of continuous functions. It is easy to see that such an embedding cannot be natural, i.e. we do not identify an element of $L^2D(U)$ with is continuous representant.

2.3 Weighted kernel of Malyshev type

Now let us consider a measurable and almost everywhere positive function $\mu : U \to \mathbb{R}$. Such a function will be called **a weight**. By $L^2(U, \mu)$ we will mean a Hilbert space of (classes of) functions, for which

$$||f||_\mu^2 := \int_U |f(w)|^2 \mu(w) \mathrm{d}w < \infty$$

with weighted inner product

$$\langle f|g\rangle_\mu := \int_U \overline{f(w)} g(w) \mu(w) \mathrm{d}w.$$

Let D be a differential operator defined on $L^2(U, \mu)$. Now we may define $L^2D(U, \mu)$ as a space of these elements from $L^2(U, \mu)$ which have continuous representant with the same inner product for which $Df = 0$ in the strong sense.

First let us recall that if D does not have smooth coefficients, then $L^2D(U)$ may not be closed. If $L^2D(U)$ is not closed, we can take the closure of it. We will use the same symbol for the closure, which should not be misleading. In what follows, we will assume that $L^2D(U)$ is already closed.

It is easy to see that if there exist constants C_1, C_2, such that $0 < C_1 < \mu < C_2$, then $L^2D(U, \mu)$ is equal as a set with $L^2D(U)$ and weighted inner product generates the same topology as classical one. Indeed, to show that we just need to write simple inequality

$$C_1 \int_U |f(w)|^2 \mathrm{d}w \leq \int_U |f(w)|^2 \mu(w) \mathrm{d}w \leq C_2 \int_U |f(w)|^2 \mathrm{d}w.$$

Consequently, $L^2D(U,\mu)$ for μ bounded from above and below by non-zero constants is closed, i.e. is a Hilbert space.

If a weight is not bounded from below or from above by non-zero constant, then the topology of $L^2D(U,\mu)$ may be different than the topology of $L^2D(U)$ and the spaces may be different as sets. Nevertheless,

Proposition 2.4. *Let coefficients of operator D be of class C^∞. Suppose that for any compact set $X \subset U$ there exists C_X, such that for any $f \in L^2D(U,\mu)$ and any $z \in X$ we have*

$$|f(z)| \leq C_X ||f||_\mu. \tag{2.2}$$

Then the space $L^2D(U,\mu)$ is a closed subspace of $L^2(U,\mu)$.

We will need the following lemma:

Lemma 2.1. *Let D be an elliptic operator. (Here we do not assume that its coefficients are of class C^∞.) Let $\{f_n\}$ be a sequence of functions, such that $Df_n = 0$ for any n which converges locally uniformly on U to some function f. Then $Df = 0$ and for any compact set $X \subset U$ $f \in L^2D(X)$.*

Proof of the Lemma: Since $f_n \to f$ locally uniformly,

$$\int_X |f_n(w) - f(w)|^2 \mathrm{d}w \leq L(X) \sup_{w \in X} |f_n(w) - f(w)|^2 \to 0,$$

where X is any compact subset of U and $L(X)$ is Lebesgue measure of X. By Proposition 2.3 $Df = 0$. Using the fact that a compact set $X \subset U$ can be chosen arbitrarily and the fact that D is a local operator ends the proof. ■

Roughly speaking, this lemma states that locally uniform limit of weak solutions of an elliptic equation is a weak solution of the same elliptic equation. If coefficients of D are smooth, then in fact this proves that locally uniform limit of strong solutions of elliptic equation is also a strong solution of the same equation.

Proof of the Proposition: Let $\{f_n\} \subset L^2D(U,\mu)$ and suppose that $f_n \to f$ in the $L^2(U,\mu)$ topology. By (2.2) f_n converges to f locally uniformly. Using Lemma 2.1 and Proposition 2.3 completes the proof. ■

From now on, D will mean such an operator for which the reproducing kernel of $L^2D(U)$ exists, e.g. an elliptic operator. Moreover by K_μ we will denote reproducing kernel of $L^2D(U,\mu)$ without further reminding.

Definition 2.6. *Let μ be a weight on U. We will say that it is **admissible** for D, if for any compact set $X \subset U$ there exists C_X, such that for any $f \in L^2D(U,\mu)$ and any $z \in C_X$ we have*

$$|f(z)| \leq C_X||f||_\mu.$$

Note that by Theorem 1.1 this condition implies the existence of a reproducing kernel and if coefficients of D are of class C^∞ by the Proposition 2.4 this condition implies that $L^2D(U,\mu)$ is closed in $L^2(U,\mu)$.

If $L^2D(U,\mu)$ is not closed in $L^2(U,\mu)$, we can take the closure and use the same symbol to denote it. As in the case of $L^2D(U)$, it should not be misleading.

Proposition 2.5. *Let $\mu > C > 0$ a.e. Then μ is admissible.*

Proof: By (2.1) for any compact set $X \subset U$ there exists $C_X > 0$, such that for any $f \in L^2D(U,\mu) \subset L^2D(U)$ we have

$$|f(z)|^2 \leq C_X \int_U |f(w)|^2 \mathrm{d}w.$$

Of course

$$\int_U |f(w)|^2 \mathrm{d}w = \frac{1}{C}\int_U |f(w)|^2 C \mathrm{d}w \leq \frac{1}{C}\int_U |f(w)|^2 \mu(w) \mathrm{d}w. \blacksquare$$

Theorem 2.5. *Let μ_1, μ_2 be weights on U, such that μ_1 is admissible and $\mu_2 \geq \mu_1$ a.e. Then μ_2 is also admissible.*

Proof: If μ_1 is admissible, then for any compact set $X \subset U$ there exists C_X, such that for any $z \in X$ and any $f \in L^2D(U, \mu_1)$

$$|f(z)| \leq C_X ||f||_{\mu_1}.$$

Since

$$\int_U |f(w)|^2 \mu_1(w) \mathrm{d}w \leq \int_U |f(w)|^2 \mu_2(w) \mathrm{d}w,$$

we have that $L^2D(U, \mu_2) \subset L^2D(U, \mu_1)$ and that for any $f \in L^2D(U, \mu_2)$

$$|f(z)| \leq C_X ||f||_{\mu_2}. \blacksquare$$

In particular, if μ is an admissible weight, then also e^μ and μ^μ are admissible weights, because $e^x > x$ and $x^x > x$ almost everywhere on the interval $[0, +\infty[$.

Corollary 2.1. *Let μ_1, μ_2 be weights on U and let μ_1 be admissible. Then $\mu_1 + \mu_2$ is also an admissible weight. In particular sum of admissible weights on the same domain is an admissible weight.*

Theorem 2.6. *Let μ_1, μ_2 be admissible weights on U, such that $\mu_2 \geq C > 0$ a.e. Then $\mu_1 \cdot \mu_2$ is an admissible weight.*

Proof: If μ_1 is admissible, then for any compact set $X \subset U$ there exists C_X, such that for any $z \in X$ and any $f \in L^2D(U, \mu_1)$

$$|f(z)| \leq C_X ||f||_{\mu_1}.$$

Since

$$\int_U |f(w)|^2 \mu_1(w) \mathrm{d}w = \frac{1}{C} \int_U |f(w)|^2 \mu_1(w) C \mathrm{d}w \leq \frac{1}{C} \int_U |f(w)|^2 \mu_1(w) \mu_2(w) \mathrm{d}w,$$

we have that $L^2D(U, \mu_1\mu_2) \subset L^2D(U, \mu_1)$ and for any $f \in L^2D(U, \mu_1\mu_2)$

$$|f(z)| \leq C_X \frac{1}{\sqrt{C}} ||f||_{\mu_1\mu_2}. \blacksquare$$

Corollary 2.2. *Let μ be an admissible weight on U and let $\alpha > 0$. Then $\alpha\mu$ is also an admissible weight.*

In particular cases we can give direct formula for a weighted kernel of Malyshev type by means of a classical one:

Proposition 2.6. *Let $\mu \equiv c > 0$ be a weight on U. Then*

$$K_\mu(z,w) = \frac{1}{c} K_1(z,w),$$

where K_1 is the reproducing kernel of $L^2D(U) = L^2D(U,1)$.

Proof: Since $\overline{K_1(z,\cdot)} \in L^2D(U)$, then it is also true that $\frac{1}{c}\overline{K_1(z,\cdot)} \in L^2D(U)$. Of course

$$\left\langle c^{-1}\overline{K_1(z,\cdot)} | f \right\rangle_c = \left\langle \overline{K_1(z,\cdot)} | f \right\rangle = f(z),$$

so the considered function K_μ has the reproducing property. ∎

It is well-known that in the case of weighted Bergman kernels more general formula occurs. In that case of square-integrable and holomorphic functions, if only weight is a square of modulus of some function f, which is holomorphic and non-zero on $\overline{U}$, then

$$K_{|f|^2}(z,w) = \frac{1}{f(z)} \frac{1}{\overline{f(w)}} K_1(z,w).$$

One would suppose that for Malyshev kernels we have something similiar — just with the change of hypothesis 'a weight is a square of modulus of some holomorphic function' with 'a weight is a square of some function from $L^2D(U)$'. This, however, is not true, because product of two solutions of given differential equation does not have to be a solution of the same equation, while product of two holomorphic functions is a holomorphic function.

2.4 Particular case: Hilbert space of harmonic functions

As we saw in the previous section, if only weight μ is bounded from below by positive constant, then there exists reproducing kernel of $L^2D(\Omega,\mu)$. In particular case, when D is Laplace's operator, we can tell much more. So until the end of section 'Particular case...' D will mean the Laplace's operator.

Note that when D is the Laplace's operator a lot is known (see e.g. [Axler2001], [Kang2001], [Ramey1996], [Koo2005]). Kernel of $L^2D(\Omega, 1)$ for Ω being the unit ball in $\mathbb{R}^n$ is equal to:

$$K(z,w) = \frac{1}{nV(B)(1-2\langle z|w\rangle + |z|^2|w|^2)^{\frac{n}{2}}} \left(\frac{n(1-|z|^2|w|^2)^2}{1-2\langle z|w\rangle + |z|^2|w|^2} - 4|z|^2|w|^2 \right),$$

where $V(B)$ is volume of n-dimensional unit ball.

Note that volume of n-dimensional ball of radius r is equal to

$$\frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}. \tag{2.3}$$

It is a classical result. See e.g. [Gipple2014] for more details.

By $L^2D(U,\mu)$ we will mean the Hilbert space of harmonic functions f defined on U, which are square-integrable in the sense

$$\int_U |f(w)|^2 \mu(w) \mathrm{d}w < \infty,$$

with the inner product

$$\langle f|g\rangle_\mu := \int_U |f(w)|^2 \mu(w) \mathrm{d}w.$$

2.4.1 Sufficient condition for a weight to be admissible

As in the case of classical Bergman space of holomorphic functions (see [Pasternak1992]), the following sufficient condition for existence of reproducing kernel of $L^2D(\Omega,\mu)$ holds:

Theorem 2.7. *Let μ be a weight on Ω, such that there exists $a > 0$, for which*

$$\int_\Omega \frac{1}{\mu^a} < \infty.$$

Then μ is admissible weight for the Laplace's operator.

Proof: Let $z \in \Omega$ be fixed. Let $r > 0$ be sufficiently small for a ball $B(z,r) := \{w \in \mathbb{R}^N | |w - z| < r\}$ to lie in Ω. Let $p := \frac{1+a}{a}$ and $q := 1 + a$. Let now $f \in L^2D(\Omega, \mu)$. Then by Mean Value Theorem for subharmonic functions we have

$$|f(z)|^{\frac{2}{p}} \leq \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}} r^n} \int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mathrm{d}w.$$

(See (2.3).)

Of course

$$\int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mathrm{d}w = \int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mu(w)^{\frac{1}{p}} \mu(w)^{-\frac{1}{p}} \mathrm{d}w.$$

Since $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we may use Hölder's inequality:

$$\int_{B(z,r)} |f(w)|^{\frac{2}{p}} \mu(w)^{\frac{1}{p}} \mu(w)^{-\frac{1}{p}} \mathrm{d}w \leq \left(\int_{B(z,r)} |f(w)|^2 \mu(w) \mathrm{d}w \right)^{\frac{1}{p}} \left(\int_{B(z,r)} \mu(w)^{-\frac{q}{p}} \mathrm{d}w \right)^{\frac{1}{q}}.$$

So we have

$$|f(z)| \leq \left(\int_{B(z,r)} \mu(w)^{-\frac{q}{p}} \mathrm{d}w \right)^{\frac{p}{2q}} \left(\frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \right)^{-\frac{p}{2}} ||f||_\mu.$$

Finally

$$|f(z)| \leq \left(\int_{B(z,r)} \mu(w)^{-a} \mathrm{d}w \right)^{\frac{1}{2a}} \left(\frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \right)^{-\frac{1+a}{2a}} ||f||_\mu. \tag{2.4}$$

Note that if $z \in B(z_0, r) \subset B(z_0, 2r) \subset \Omega$, then $B(z,r) \subset B(z_0, 2r)$ and in fact we have

$$|f(z)| \leq \left(\int_{B(z_0,2r)} \mu(w)^{-a} \mathrm{d}w \right)^{\frac{1}{2a}} \left(\frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \right)^{-\frac{1+a}{2a}} ||f||_\mu$$

for any $z \in B(z_0, r)$, so the weight is admissible. ■

2.4.2 An example of a weight which is not admissible

An example of weight for which there is no reproducing kernel of corresponding weighted Bergman space was found by Z. Pasternak-Winiarski (see [Pasternak1992]). Here we will use similiar idea to give example of non-admissible weight for $L^2D(\Omega, \mu)$. We will need the following theorem by Runge (see [Rudin1974] for more details):

Theorem 2.8. *Let $X \subset \mathbb{C}$ be a compact set, such that $\mathbb{C} \setminus X$ is connected. Let $f : X \to \mathbb{C}$ be continuous on X and holomorphic on interior of X. Then f is a uniform limit of a sequence of holomorphic polynomials on X.*

Let Ω be unit disk in $\mathbb{R}^2$. Let

$$A_n := \{(x,y) \in \mathbb{R}^2 : ||(x,y)|| < 2^{-n}\} \cup \{(x,y) \in \mathbb{R}^2 : |y| < 2^{-n} \wedge 0 < x < 1\},$$

where

$$|| \cdot ||$$

is classical norm on $\mathbb{R}^2$. Let

$$M_n := (\overline{\Omega} \setminus A_n) \cup \overline{A_{n+1}}.$$

Now let $f_n : M_n \to \mathbb{R}^2$ be defined in the following way

$$f_n(x,y) := \begin{cases} 1 + \frac{1}{n} & \text{for} \quad (x,y) \in \overline{A_{n+1}} \\ 0 & \text{for} \quad (x,y) \in \overline{\Omega} \setminus (A_n \cup B_n), \end{cases}.$$

where $B_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \wedge |y| < 2^{-n}\}$. By Theorem 2.8 there exist holomorphic polynomials G_n, such that

$$|G_n(x,y) - f_n(x,y)| < \frac{1}{n}$$

for any $(x,y) \in M_n$. Bearing in mind that a sequence of holomorphic functions is convergent if and only if its real and imaginary part are convergent and imaginary part of f_n is zero, we conclude that in fact there exist harmonic polynomials g_n, such that

$$|g_n(x,y) - f_n(x,y)| < \frac{1}{n}$$

for any $(x,y) \in M_n$. It implies that $|g_n(x,y)| < \frac{1}{n}$ for $(x,y) \in \overline{\Omega} \setminus A_n$ and $1 < |g_n(x,y)| < 1 + \frac{2}{n}$ for $(x,y) \in \overline{A_{n+1}}$. Now let us define polynomials:

$$h_n(x,y) := \frac{g_n(x,y)}{g_n(0,0)}.$$

Since $|g_n(0,0)| > 1$, h_n is well-defined. Moreover

$$\left(1 + \frac{2}{n}\right)^{-1} < |h_n(x,y)| < 1 + \frac{2}{n}$$

on $\overline{A_{n+1}}$ and

$$|h_n(x,y)| < \frac{1}{n}$$

on $\overline{\Omega} \setminus A_n$. Now let us denote

$$D_n := \Omega \cap \overline{A_n}.$$

Then we may define a weight:

$$\mu_n(x,y) := \begin{cases} 1 & \text{if} \quad (x,y) \in \Omega \setminus D_1 \\ 0 & \text{if} \quad x = 1 \\ \min\left\{1, \frac{1}{|h_n(x,y)|^2}\right\} & \text{if} \quad (x,y) \in D_n \setminus D_{n+1}. \end{cases}$$

Since μ is bounded from above (by 1), $h_n \in L^2D(\Omega, \mu)$ for any $n \in \mathbb{N}$, as harmonic polynomials. It is easy to show that

$$|h_n(x,y)|^2 \mu(x,y) < 9$$

and

$$\lim_{n\to\infty} |h_n(x,y)|^2 \mu(x,y) = 0.$$

Therefore, by Lebesgue Majorized Convergence Theorem we have

$$\int_\Omega \lim_{n\to\infty} |h_n(x,y)|^2 \mu(x,y) \mathrm{d}w = \lim_{n\to\infty} \int_\Omega |h_n(x,y)|^2 \mu(x,y) \mathrm{d}w = 0.$$

By its own definition, $|h_n(0,0)| = 1$ for any $n \in \mathbb{N}$, but $||h_n||_\mu \to 0$. It means that functional of point evaluation $L^2D(\Omega, \mu) \ni f \mapsto f(0,0) \in \mathbb{R}$ is not continuous and by Theorem 1.1 reproducing kernel of $L^2D(\Omega, \mu)$ does not exist.

2.5 Dependence of Malyshev kernel on a weight of integration

Dependence of two canonical kernels (of Bergman type and of Szegö type) on weight of integration was widely investigated. Z. Nehari (see [Nehari1952]) considered the case of continuous weights. Z. Pasternak-Winiarski (see [Pasternak1990]) showed that the Bergman kernel depends in analytic way on a weight of integration, without hypothesis that weights need to be continuous. He, however, assumed that weights converge in pretty strong topology.

In this dissertation it will be proved that kernels of Malyshev type depend 'only' in continuous way on a weight of integration. We will do this, however, with very weak assumptions — weights will just need to be measurable functions and they will need to converge to a limit weight only almost everywhere.

Now we assume again that D is an elliptic operator.

Theorem 2.9. *Let $\{\mu_n\}$ be a sequence of weights on U convergent almost everywhere to μ, such that there exist $C_1, C_2 > 0$, for which $C_1 < \mu < C_2$ and $C_1 < \mu_n < C_2$ a.e. for each $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} K_{\mu_n},$$

where the limit above is locally uniform on $U \times U$, exists and is equal to K_μ.

By $K(z,w)$ we will mean pointwise limit of $K_{\mu_n}(z,w)$.

Lemma 2.2. *Let $\{\mu_n\}$ be a sequence of weights on U convergent almost everywhere to μ, such that $\mu > c > 0$ a.e. Let locally uniform limit of K_{μ_n} exist on U. Then for any $z \in U$ the following conditions are equivalent:*

(i) $K_{\mu_n}(z,z) \to K_\mu(z,z)$ for almost every $z \in U$;

(ii) $K_{\mu_n}(z,\cdot) \to K_\mu(z,\cdot)$ locally uniformly on $U \times U$.

Proof: We need only to show implication $(i) \Rightarrow (ii)$.

By Lemma 2.1, locally uniform limit K is an element of kernel of operator D. By Fatou's Lemma and our assumptions:

$$\begin{aligned}\int_U |K(z,w)|^2 \mu(w) \mathrm{d}w &\le \liminf_{n\to\infty} \int_U |K_{\mu_n}(z,w)|^2 \mu_n(w) \mathrm{d}w \\ &= \liminf_{n\to\infty} K_{\mu_n}(z,z) = K_\mu(z,z).\end{aligned}$$

We showed that $K(z,\cdot) \in L^2D(U,\mu)$. Moreover

$$\left|\left|\overline{K(z,\cdot)}\right|\right|_\mu \le \sqrt{K_\mu(z,z)}$$

and if only $K_\mu(z,z) > 0$,

$$\left\| \frac{\overline{K(z,\cdot)}}{K_\mu(z,z)} \right\|_\mu \le \frac{1}{\sqrt{K_\mu(z,z)}},$$

so by Theorem 1.3,

$$\frac{K(z,w)}{K_\mu(z,z)} = \frac{K_\mu(z,w)}{K_\mu(z,z)},$$

which means that $K(z,w) = K_\mu(z,w)$.

If $K_\mu(z,z) = 0$, then by Theorem 1.2, we have also $K(z,\cdot) \equiv 0$ and $K_\mu(z,\cdot) \equiv 0$, so $K(z,\cdot) = K_\mu(z,\cdot)$. ■

Lemma 2.3. *Let μ_1, μ_2 be weights on U, such that $0 < c < \mu_1 \le \mu_2$ a.e. Then for any $z \in U$ we have*

$$K_{\mu_2}(z,z) \le K_{\mu_1}(z,z).$$

Proof: First let us assume that $K_{\mu_1}(z,z)$ and $K_{\mu_2}(z,z)$ are greater than 0. By Theorem 1.3 it is true that

$$\frac{1}{K_{\mu_1}(z,z)} = \int_U \left| \frac{K_{\mu_1}(z,w)}{K_{\mu_1}(z,z)} \right|^2 \mu_1(w)\mathrm{d}w \le \int_U \left| \frac{K_{\mu_2}(z,w)}{K_{\mu_2}(z,z)} \right|^2 \mu_1(w)\mathrm{d}w.$$

Since $\mu_1 \le \mu_2$,

$$\int_U \left| \frac{K_{\mu_2}(z,w)}{K_{\mu_2}(z,z)} \right|^2 \mu_1(w)\mathrm{d}w \le \int_U \left| \frac{K_{\mu_2}(z,w)}{K_{\mu_2}(z,z)} \right|^2 \mu_2(w)\mathrm{d}w.$$

Because

$$\int_U \left| \frac{K_{\mu_2}(z,w)}{K_{\mu_2}(z,z)} \right|^2 \mu_2(w)\mathrm{d}w = \frac{1}{K_{\mu_2}(z,z)},$$

in conclusion we have that

$$\frac{1}{K_{\mu_1}(z,z)} \le \frac{1}{K_{\mu_2}(z,z)},$$

which ends the proof.

Now let $K_{\mu_1}(z,z) = 0$. Then by Theorem 1.2 we have $K_{\mu_1}(z,\cdot) \equiv 0$. Since $\mu_1 \le \mu_2$, we have $\overline{K_{\mu_2}(z,\cdot)} \in L^2D(U,\mu_1)$. Then by Theorem 1.2 again we have $K_{\mu_2}(z,\cdot) \equiv 0$, so $K_{\mu_2}(z,z) \le K_{\mu_1}(z,z)$.

If $K_{\mu_2}(z,z) = 0$, then of course $K_{\mu_2}(z,z) \le K_{\mu_1}(z,z)$. ■

Proof of the Main Theorem: Let $X \subset U$ be any compact set. We have

$$\int_X |K_{\mu_n}(z,w)|^2 \mathrm{d}w \le \int_U |K_{\mu_n}(z,w)|^2 \mathrm{d}w.$$

Moreover

$$\frac{1}{C_1}\int_U |K_{\mu_n}(z,w)|^2 C_1 \mathrm{d}w \le \frac{1}{C_1}\int_U |K_{\mu_n}(z,w)|^2 \mu_n(w) \mathrm{d}w = \frac{1}{C_1} K_{\mu_n}(z,z).$$

By Lemma 2.3,

$$K_{\mu_n}(z,z) \le K_{\frac{C_1}{2}}(z,z),$$

where $K_{\frac{C_1}{2}}$ denotes the reproducing kernel of $L^2D(U, \frac{C_1}{2})$. It means that sequence $\{K_{\mu_n}(z,\cdot)\}$ is bounded in $L^2D(U)$. By Theorem 2.2 we claim that $\{K_{\mu_n}(z,\cdot)\}$ is bounded also in the Sobolev space $W^{2,2}(X)$. Now, by Theorem 2.1, we see that $\{K_{\mu_n}(z,\cdot)\}$ is also bounded in the Hölder's space $C^{0,\gamma}(X)$ for any $\gamma > 0$. This means that the hypotheses of the Arzelá-Ascoli Theorem are satisfied and in our sequence $\{K_{\mu_n}(z,\cdot)\}$ there exists a subsequence which is locally uniformly convergent to some function K. Without losing generality we may identify such a convergent subsequence with whole sequence.

We need only to show that K is the reproducing kernel of $L^2D(U,\mu)$. We will divide our prove into two cases.

case 1: Let $K_\mu(z,z) = 0$. Then by Theorem 1.2 also $K_\mu(z,\cdot) \equiv 0$. In addition, since $L^2D(U,\mu)$ is equal as a set with $L^2D(U,1)$, we have $\overline{K(z,\cdot)} \in L^2D(U,1)$ and again by Theorem 1.2 $K(z,\cdot) \equiv 0$. So we showed that $K_\mu(z,\cdot) = K(z,\cdot)$.

case 2: Let $K_\mu(z,z) > 0$. Since all μ_n and μ are uniformly bounded from below and above by non-zero constants, all spaces $L^2D(U,\mu_n)$ and $L^2D(U,\mu)$ are pairwise equal as sets. By Proposition 1.3 for any $f \in L^2D(U,\mu_n)$

$$|f(z)| \le \sqrt{K_{\mu_n}(z,z)}||f||_{\mu_n}.$$

Taking limit we get

$$|f(z)| \le \sqrt{K(z,z)}||f||_{\mu},$$

where Lebesgue's Dominated Convergence Theorem can be used to show that $||f||_{\mu_n} \to ||f||_\mu$. For $f(\cdot) := \overline{K_\mu(z,\cdot)}$, we obtain

$$K_\mu(z,z) \leq \sqrt{K(z,z)}\sqrt{K_\mu(z,z)}$$

and in consequence

$$K_\mu(z,z) \leq K(z,z).$$

So also $K(z,z) > 0$.

By Fatou's Lemma and our assumptions:

$$\int_U |K(z,w)|^2 \mu(w) \mathrm{d}w \leq \liminf_{n\to\infty} \int_U |K_{\mu_n}(z,w)|^2 \mu_n(w) \mathrm{d}w =$$
$$\liminf_{n\to\infty} K_{\mu_n}(z,z) = K(z,z).$$

Therefore by Lemma 2.1 $\overline{K(z,\cdot)} \in L^2D(U,\mu)$ and

$$\left|\left|\frac{\overline{K(z,\cdot)}}{K(z,z)}\right|\right|_\mu \leq \frac{1}{\sqrt{K(z,z)}}.$$

By Theorem 1.3, $K(z,z) \leq K_\mu(z,z)$.

So we showed that $K_\mu(z,z) = K(z,z)$. By Lemma 2.2 K_{μ_n} converges locally uniformly to K_μ. ■

2.6 Ramadanov Theorem for Malyshev Kernels

A version of continuous dependence of general reproducing kernels on increasing or decreasing sequences of domains can be found in a seventy-year-old paper of N. Aronszajn (see [Aronszajn1950]). In 1967, I. Ramadanov published his famous research ([Ramadanov1967]) in which he considered the continuous dependence of the classical Bergman kernel on an increasing sequence of domains. His results were obtained later in a different way by M. Skwarczyński ([Skwarczyński1985a], [Skwarczyński1985b]). As we will see in this dissertation, Skwarczyński's and Ramadanov's techniques can be slightly changed to prove continuous dependence of the kernel defined by V. M. Malyshev on an increasing sequence of domains. Moreover in the section 'One more proof of the Ramadanov Theorem' we will give a new proof of the Ramadanov Theorem, using weak convergence.

In whole this section we will assume without further reminding that differential operator D is well-defined on a sum of all considered domains.

2.6.1 Case of an Increasing Sequence of Domains

The aim of this section is to prove the following:

Theorem 2.10. *Let D be an elliptic operator. Let $\{U_n\}$ be an increasing sequence of domains and $U = \bigcup_{n=1}^{+\infty} U_n$. Then*

$$\lim_{n\to\infty} K_{U_n}(z,w)$$

exists and is equal to $K_U(z,w)$, where the limit above is locally uniform on $U \times U$.

In the remainder of this section, $K(z,w)$ will mean the pointwise limit of $K_{U_n}(z,w)$.

Before giving the proof of the Main Theorem, we will prove some lemmas.

Lemma 2.4. *With hypotheses as in the Theorem above, we have*

$$K_{U_{n+1}}(z,z) \le K_{U_n}(z,z)$$

for any $z \in U_n$. Moreover,

$$K_U(z,z) \le K_{U_n}(z,z)$$

for any $n \in \mathbb{N}$, which in the limit becomes

$$K_U(z,z) \le K(z,z).$$

Proof: First let us assume that $K_{U_n}(z,z) > 0$ and $K_{U_{n+1}}(z,z) > 0$. By the reproducing property and Theorem 1.3

$$\frac{1}{K_{U_n}(z,z)} = \int_{U_n} \left| \frac{\overline{K_{U_n}(z,w)}}{K_{U_n}(z,z)} \right|^2 \mathrm{d}w \le \int_{U_n} \left| \frac{\overline{K_{U_{n+1}}(z,w)}}{K_{U_{n+1}}(z,z)} \right|^2 \mathrm{d}w.$$

By properties of the integral and again by the reproducing property,

$$\int_{U_n} \left| \frac{\overline{K_{U_{n+1}}(z,w)}}{K_{U_{n+1}}(z,z)} \right|^2 \mathrm{d}w \le \int_{U_{n+1}} \left| \frac{\overline{K_{U_{n+1}}(z,w)}}{K_{U_{n+1}}(z,z)} \right|^2 \mathrm{d}w = \frac{1}{K_{U_{n+1}}(z,z)}.$$

In order to prove the second part of the Lemma for the case of $K_U(z,z) > 0$ and $K_{U_n}(z,z) > 0$ for any $n \in \mathbb{N}$, we just need to swap $K_{U_{n+1}}(z,z)$ with $K_U(z,z)$ in the considerations above.

Now let us assume that $K_{U_n}(z,z) = 0$ for some $n \in \mathbb{N}$. Then, for $m > n$, also $K_{U_m}(z,z) = 0$ and $K_U(z,z) = 0$. Indeed, if $K_{U_n}(z,z) = 0$ then by Theorem 1.2 for any $f \in L^2D(U_n)$ we have $f(z) = 0$. But if $g \in L^2D(V)$ for $V \supset U_n$, then g also is an element of $L^2D(U_n)$, so for any $g \in L^2D(V)$ we have $g(z) = 0$. By Theorem 1.2 again $K_{U_m}(z,z) = 0$ and $K_U(z,z) = 0$.

If $K_U(z,z) = 0$, then of course $K_U(z,z) \le K_{U_n}(z,z)$ for any $n \in \mathbb{N}$ and $K_U(z,z) \le K(z,z)$. ■

Lemma 2.5. *Let $U_1, U_2, \ldots$ be a sequence of domains and U be a limit domain in the sense of pointwise limit of the sequence of indicator functions of sets $U_1, U_2, \ldots$. Let f be an arbitrary positive-valued function defined and integrable on $\bigcup_{n=1}^{+\infty} U_n$. Then*

$$\lim_{n\to\infty} \int_{U_n} f(w)\mathrm{d}w = \int_U f(w)\mathrm{d}w.$$

Proof: Let χ_X be the indicator function of set X. Then of course $f(w)\chi_{U_n}(w) \le f(w)$ and, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{U_n} f(w)\mathrm{d}w = \lim_{n\to\infty} \int_U f(w)\chi_{U_n}(w)\mathrm{d}w$$
$$= \int_U \lim_{n\to\infty} f(w)\chi_{U_n}(w)\mathrm{d}w = \int_U f(w)\mathrm{d}w. \blacksquare$$

Lemma 2.6. *Let $U_1, U_2, \ldots,$ be a sequence of domains convergent to a domain U in the sense of pointwise convergence of the sequence of indicator functions of sets U_n, which satisfies the following condition:*

(A) For each compact set $X \subset U$ there exists m such that for $n > m$, $X \subset U_n$.

Let locally uniform limit of K_{U_n} exist on U. Then the following conditions are equivalent.

(i) $K_{U_n}(z,z) \to K_U(z,z)$ *for almost every* $z \in U$*;*

(ii) $K_{U_n} \to K_U$ *locally uniformly on* $U \times U$*.*

Note that, if $\{U_n\}$ is an increasing sequence of domains or if $U \subset U_n$ for large enough n, then condition (A) is satisfied. Moreover, it is easy to see that in the case of an increasing or decreasing sequence of domains $\bigcup_{n=1}^{+\infty} U_n$ and $\bigcap_{n=1}^{+\infty} U_n$, are equivalent to the limits introduced in the Lemma.

Proof of the Lemma: We need only prove the implication $(i) \Rightarrow (ii)$.

Let $X \subset U$ be compact. Then there exists $m \in \mathbb{N}$, such that, for any $n > m$, $X \subset U_n$. By Fatou's lemma:

$$\int_X |K(z,w)|^2 \mathrm{d}w \le \liminf_{n\to\infty} \int_X |K_{U_n}(z,w)|^2 \mathrm{d}w \le \liminf_{n\to\infty} \int_{U_n} |K_{U_n}(z,w)|^2 \mathrm{d}w$$
$$= \liminf_{n\to\infty} K_{U_n}(z,z) = K_U(z,z).$$

In conclusion $||\overline{K(z,\cdot)}||_U \le \sqrt{K_U(z,z)}$ and if only $K_U(z,z) > 0$, we have

$$\left|\left|\frac{\overline{K(z,\cdot)}}{K_U(z,z)}\right|\right|_U \le \frac{1}{\sqrt{K_U(z,z)}}.$$

By Lemma 2.1, $\overline{K(z,\cdot)} \in L^2D(U)$. By Theorem 1.3

$$K(z,\cdot) = K_U(z,\cdot),$$

which ends the proof.

On the other hand, if $K_U(z,z) = 0$, then by Theorem 1.2 also $K_U(z,\cdot) \equiv 0$ and $K(z,\cdot) \equiv 0$, so $K_U(z,\cdot) = K(z,\cdot)$. ■

Proof of the Main Theorem: Let $X \subset U$ be any compact set, $m, n \in \mathbb{N}$ be such that $X \subset U_n$ for any $n \ge m$. We have

$$||K_{U_n}(z,\cdot)||_X^2 \le ||K_{U_n}(z,\cdot)||_U^2 = K_{U_n}(z,z).$$

By Lemma 2.4,

$$0 \leq K_{U_{n+1}}(z,z) \leq K_{U_n}(z,z)$$

which means that the sequence $\{K_{U_n}(z,\cdot)\}$ is bounded in $L^2D(U)$. By Theorem 2.2 we claim that $\{K_{U_n}(z,\cdot)\}$ is bounded also in the Sobolev space $W^{2,2}(X)$. Now, by Theorem 2.1, we see that $\{K_{U_n}(z,\cdot)\}$ is also bounded in the Hölder's space $C^{0,\gamma}(\overline{X})$ for any $\gamma > 0$. This means that the hypotheses of the Arzelá-Ascoli theorem are satisfied and in our sequence $\{K_{U_n}(z,\cdot)\}$ there exists a subsequence which is locally uniformly convergent to some function K. We need only show that the limit of such a convergent subsequence is the reproducing kernel of the indicated space. Without loss of generality, we may identify such a convergent subsequence with the whole sequence.

Note that $\lim_{n\to\infty} K_{U_n}(z,z)$ must exist, because the sequence $\{K_{U_n}(z,z)\}$ is bounded and monotonic.

By condition (A), there exists m such that $X \subset U_n$ for $n > m$. Then, by Fatou's Lemma and our assumptions,

$$\int_X |K(z,w)|^2 \mathrm{d}w \leq \liminf_{n\to\infty} \int_X |K_{U_n}(z,w)|^2 \mathrm{d}w \leq \liminf_{n\to\infty} \int_{U_n} |K_{U_n}(z,w)|^2 \mathrm{d}w.$$

But

$$\liminf_{n\to\infty} \int_{U_n} |K_{U_n}(z,w)|^2 \mathrm{d}w = \liminf_{n\to\infty} K_{U_n}(z,z) = K(z,z). \tag{2.5}$$

By Lemma 2.1 and the arbitrariness of the choice of compact set X we have $\overline{K(z,\cdot)} \in L^2D(U)$. Now we need to show that $K(z,\cdot) = K_U(z,\cdot)$. We will consider two cases.

Case 1: First, let $K_U(z,z) = 0$ for some $z \in U$. Then, by Theorem 1.2, also $K(z,z) = 0$, as the value of $\overline{K(z,\cdot)}$ at the point z. By (2.5), $K(z,\cdot) \equiv 0$ on U, since $|K(z,\cdot)|^2$ is continuous and non-negative. Using again Theorem 1.2, we conclude that $K(z,\cdot) = K_U(z,\cdot)$.

Case 2: Now let $K_U(z,z) > 0$. Then, by Lemma 2.4, we also have $K(z,z) > 0$. Moreover, by (2.5),

$$||\overline{K(z,\cdot)}||_U \leq \sqrt{K(z,z)}$$

and

$$\left\| \frac{\overline{K(z,\cdot)}}{K(z,z)} \right\|_U \leq \frac{1}{\sqrt{K(z,z)}}.$$

By Theorem 1.3, $K(z,z) \leq K_U(z,z)$.

Of course $L^2D(U) \subset L^2D(U_{n+1}) \subset L^2D(U_n)$ for any $n \in \mathbb{N}$. Therefore by 1.3, for $f \in L^2D(U)$, we can write inequality (1.1) in the following form

$$|f(z)| \leq \sqrt{K_{U_n}(z,z)} \sqrt{\int_{U_n} |f(w)|^2 \mathrm{d}w}\,. \tag{2.6}$$

Taking the limit in (2.6) and using Lemma 2.5 we get

$$|f(z)| \leq \sqrt{K(z,z)} \sqrt{\int_U |f(w)|^2 \mathrm{d}w}.$$

In particular, for $f(\cdot) := \overline{K_U(z,\cdot)} \in L^2D(U)$ we have

$$|K_U(z,z)| \leq \sqrt{K(z,z)} \sqrt{\int_U |K_U(z,w)|^2 \mathrm{d}w} = \sqrt{K(z,z)}\sqrt{K_U(z,z)},$$

so $K_U(z,z) \leq K(z,z)$.

Finally, $K_U(z,z) = K(z,z)$ and using Lemma 2.6 ends the proof. ■

2.6.2 Case of a Decreasing Sequence of Domains

First, we will define $D(U)$ as a set of these functions f from $L^2D(U)$, for which there exists a domain $V \supset \overline{U}$, such that $f \in L^2D(V)$.

Theorem 2.11. *Let D be an elliptic operator. Let $\{U_n\}$ be a decreasing sequence of domains and $U = \bigcap_{n=1}^{+\infty} U_n$ be a bounded domain. If $D(U)$ is dense in $L^2D(U)$, then*

$$\lim_{n\to\infty} K_{U_n}(z,w)$$

exists and is equal to $K_U(z,w)$, where the limit above is locally uniform on $U \times U$.

As in the preceding section, we will denote the pointwise limit of $K_{U_n}(z,w)$ by $K(z,w)$.

Proof: Let $z \in U$ and $f \in L^2D(U)$. Let $h \in D(U)$. Then, for n large enough, we have $h \in L^2D(U_n)$. By inequality (1.1) and Proposition 1.3,

$$|h(z)| \leq \sqrt{K_{U_n}(z,z)}\sqrt{\int_{U_n} |h(w)|^2 \mathrm{d}w}\,. \tag{2.7}$$

Taking the limit in (2.7) and using Lemma 2.5 we get

$$|h(z)| \leq \sqrt{K(z,z)}\sqrt{\int_{U} |h(w)|^2 \mathrm{d}w}.$$

We know that $K(z,z)$ exists because, in a fashion similar to Lemma 2.4, it can be shown that

$$K_U(z,z) \geq K_{U_{n+1}}(z,z) \geq K_{U_n}(z,z).$$

(One notable difference here is that if $K_{U_n}(z,z) = 0$ for some n, then also $K_{U_m}(z,z) = 0$ for $m < n$. Also if $K_U(z,z) = 0$, then for each $n \in \mathbb{N}$ we have $K_{U_n}(z,z) = 0$.)

In conclusion,

$$K_U(z,z) \geq K(z,z),$$

i.e. $K_{U_n}(z,z)$ is increasing and bounded from above.

On the other hand, by our assumptions there exists a sequence $\{h_n\} \subset D(U)$, such that $h_n \to f$ in the $L^2(U)$ sense. So we can write

$$|f(z)| \leq \sqrt{K(z,z)}\sqrt{\int_{\overline{U}} |f(w)|^2 \mathrm{d}w} = \sqrt{K(z,z)}\sqrt{\int_{U} |f(w)|^2 \mathrm{d}w}.$$

Taking $f(\cdot) = \overline{K_U(z,\cdot)}$ we get

$$K_U(z,z) \leq \sqrt{K(z,z)}\sqrt{K_U(z,z)}$$

and if $K_U(z,z) > 0$, then

$$K_U(z,z) \leq K(z,z).$$

If $K_U(z,z) = 0$, then also $K(z,z) = 0$. In conclusion $K_U(z,z) = K(z,z)$.

Now let $X \subset U$ be any compact set. We have

$$||K_{U_n}(z,w)||_X^2 \leq ||K_{U_n}||_U^2 \leq K_{U_n}(z,z) \leq K_U(z,z),$$

so the sequence $\{K_{U_n}(z,\cdot)\}$ is bounded in $L^2D(U)$. By Theorem 2.2 we claim that $\{K_{U_n}(z,\cdot)\}$ is bounded also in the Sobolev space $W^{2,2}(X)$. Now by Theorem 2.1 we see that $\{K_{U_n}(z,\cdot)\}$ is also bounded in the Hölder's space $C^{0,\gamma}(\overline{X})$ for any $\gamma > 0$. This means that the hypotheses of the Arzelá-Ascoli Theorem are satisfied and in our sequence $\{K_{U_n}(z,\cdot)\}$ there exists a subsequence which is locally uniformly convergent to some function K. Using Lemma 2.6 ends the proof. ■

2.6.3 The Ramadanov Theorem and Orthogonal Projections

The main aim of this section is to prove the following theorem:

Theorem 2.12. *Let D be an elliptic operator. Let $U = \bigcup_{n=1}^{+\infty} U_n$, $U_1 \Subset U_2 \Subset U_3 \Subset \dots$. Then*

$$\lim_{n\to\infty} K_{U_n}(z,\cdot) = K_U(z,\cdot),$$

for any $z \in U$.

Note that this theorem is stronger than Theorem 2.10, because convergence here is convergence in norm and by (1.1) and Proposition 1.3 it implies locally uniform convergence. (Remember that we may extend $K_{U_n}(z,\cdot)$ to U by zero.)

In order to prove this theorem we will need Stone's Theorem and the following Lemma:

Theorem 2.13. *(Stone) Let $F_1 \Supset F_2 \Supset F_3 \dots$ be a sequence of closed subspaces of the Hilbert space H and $F = \bigcap_{n=0}^{\infty} F_n$. Let $P_i : H \to F_i$, $P : H \to F$ be orthogonal projections. Then, for any $f \in H$, we have*

$$P_n f \to Pf.$$

See [Stone1990] for more details and the proof.

Lemma 2.7. *Let $U = \bigcup_{n=1}^{+\infty} U_n$, $U_1 \Subset U_2 \Subset U_3 \Subset \dots$. Let $F_n := \{f \in L^2(U) : f\big|_{U_n} \in L^2D(U_n)\}$. Let P_n be orthonormal projection onto F_n. Then*

$$Pf = \lim_{n\to\infty} P_n f,$$

where $f \in L^2(U)$.

Proof: By definition, F_n is a closed subspace of $L^2(U)$ for any natural number n. Moreover, $f \in F_{n+1}$ implies that $f \in F_n$. Indeed,

$$\int_{U_n} |f(w)|^2 \mathrm{d}w \leq \int_{U_{n+1}} |f(w)|^2 \mathrm{d}w < \infty \tag{2.8}$$

and since $Df = 0$ on U_{n+1}, then also $Df = 0$ on U_n. So we know that $F_1 \Supset F_2 \Supset F_3 \Supset \cdot \ldots$ Of course $L^2D(U) = \bigcap_{n=1}^{+\infty} F_n$, because of the fact that $Df = 0$ on U is equivalent to $Df = 0$ on each U_n and because of inequality (2.8). Using Stone's Theorem ends the proof. ■

Proof of the Main Theorem: Let $P : L^2(U) \to L^2D(U)$ be the orthogonal projection. Then, for any $h \in L^2(U)$, we have

$$h = h_1 + h_2 \in L^2D(U) \oplus L^2D(U)^{\perp}.$$

Of course

$$\int_U h(w)K_U(z,w)dw = \int_U h_1(w)K_U(z,w)dw = h_1(z).$$

Thus we can write

$$(Ph)(z) = \int_U h(w)K_U(z,w)dw.$$

Now let $X \subset U$ be a domain and $z \in X$. Let h_z be defined in the following way:

$$h_z(w) := \begin{cases} K_X(z,w) & for \quad w \in X \\ 0 & for \quad w \in U \setminus X \end{cases}.$$

Of course such an h_z is an element of $L^2(U)$.

For any $f \in L^2D(U)$ we have

$$\langle f|Ph_z\rangle = \langle Pf|h_z\rangle = \langle f|h_z\rangle = \int_X f(w)K_X(z,w)\mathrm{d}w = f(z).$$

Since the only element in $L^2D(U)$ with the reproducing property is its reproducing kernel, we have

$$K_U(z,\cdot) = Ph_z, z \in U.$$

Similiarly, if $X \subset U_n$, we have

$$K_{U_n}(z, \cdot) = Ph_z, z \in U$$

Using the Lemma we obtain that

$$\lim_{n\to\infty} P_n h_z = Ph_z,$$

which completes the proof. ■

2.6.4 One More Proof of the Ramadanov Theorem

Theorem 2.12 can be proved in an another way. Let $f \in L^2(U)$. Then

$$\int_U \chi_{U_n}(w) K_{U_n}(z, w) f(w) \mathrm{d}w = [P_n f](z),$$

where χ_X is the indicator function of set X and P_n is the orthogonal projection of $L^2(U)$ onto $L^2D(U_n)$ as in the proof from the previous Section. By Stone's Theorem $[P_n f](z) \to [Pf](z)$, where P is the orthogonal projection of $L^2(U)$ onto $L^2D(U)$. So we have

$$\int_U \chi_{U_n}(w) K_{U_n}(z, w) f(w) \mathrm{d}w \to \int_U \chi_U(w) K_U(z, w) f(w) \mathrm{d}w.$$

Since f was chosen arbitrarily from $L^2(U)$, we conclude that $\chi_{U_n} K_{U_n}(z, \cdot)$ converges weakly to $\chi_U K_U(z, \cdot)$. Now we need only show that

$$\lim_{n\to\infty} \int_U \chi_{U_n} |K_{U_n}(z, w)|^2 \mathrm{d}w \leq \int_U \chi_U |K_U(z, w)|^2 \mathrm{d}w$$

to prove that in fact $\chi_{U_n} K_{U_n}(z, \cdot)$ converges to $\chi_U K_U(z, \cdot)$ in the strong topology of L^2, i.e. we need to show that

$$\lim_{n\to\infty} K_{U_n}(z, z) \leq K_U(z, z).$$

This can be done in the same way as in the Proof of the Main Theorem from Section 2.6.1. ■

2.7 Applications to Partial Differential Equations Theory

In fact, by using reproducing kernels theory, we solved some extremal problems for solutions of elliptic equations.

Theorem 2.14. *Let $U \subset \mathbb{R}^2$, $z \in U$, $c \in \mathbb{R}$ and weight μ be bounded from below by non-zero constant. In the set $V^{\mu}_{z,c} := \{f \in L^2(U,\mu) : Df = 0 \wedge f(z) = c\}$ of weighted square-integrable solutions of the elliptic equation $Df = 0$, for which $f(z) = c$, if it is not empty, there exists exactly one element f_0, such that*

$$||f_0||_{\mu} = \min_{f \in V^{\mu}_{z,c}} ||f||_{\mu}.$$

Such an element in what follows will be called a ***minimal*** (z,c)***-solution in weight*** *μ of the elliptic equation $Df = 0$ on U. If $\mu \equiv 1$ we will just write $V_{z,c}$ instead of $V^{\mu}_{z,c}$ and say 'minimal (z,c)-solution of elliptic equation' instead of 'minimal (z,c)-solution of elliptic equation in weight 1'.*

Of course if μ is integrable on U and $c(x) \equiv 0$ in the divergence form of an elliptic equation, then each constant function is an element of $L^2D(U,\mu)$ and therefore $V_{z,c}$ is not empty. In particular, it is true for weight $\mu \equiv 1$ and a bounded domain U.

Theorem 2.15. *Let μ_n be a sequence of weights convergent to μ a.e. on U, $z \in U$ and $c \in \mathbb{R}$. Let f_n denotes minimal (z,c)-solution in weight μ_n of the elliptic equation $Du = 0$. Let f be minimal (z,c)-solution in weight μ of the elliptic equation $Du = 0$. Then*

$$\lim_{n \to \infty} f_n = f,$$

where the limit above is locally uniform on $U \times U$.

Theorem 2.16. *Let U_n be an increasing sequence of bounded domains, $U = \bigcup_{n=1}^{+\infty} U_n$, $z \in U$, $c \in \mathbb{R}$. Let f^z_n be a minimal (z,c)-solution of the elliptic equation $Du = 0$ on U_n and let f^z be a minimal (z,c)-solution of the elliptic equation $Du = 0$ on U. Then*

$$f^z_n \to f^z$$

in the topology $L^2(U)$.

Proof of the Theorems: For $c = 1$ it is just a consequence of Theorems 1.3, 2.9 and 2.12.

Now let $c \neq 0, c \neq 1$. Then the linear operator

$$Af := cf$$

is a bijection between $V^{\mu}_{z,1}$ and $V^{\mu}_{z,c}$, and

$$||Af||_{\mu} = |c| \cdot ||f||_{\mu}.$$

Therefore in $V^{\mu}_{z,c}$ there is exactly one element f_c with minimal weighted L^2-norm and

$$f_c = cf_1,$$

where f_1 is the unique element of $V^{\mu}_{z,1}$ with minimal weighted L^2-norm.

Now let us consider the case $c = 0$. Of course zero is the only element of $V^{\mu}_{z,0}$ with minimal norm for any domain and zero is locally the uniform limit of the sequence of zero functions. ■

2.7.1 Estimates for a minimal solution of the Laplace's equation

Theorem 2.17. *Let Ω be a domain in $\mathbb{R}^n$ with a boundary of class C^1. Let μ be a weight on Ω, such that*

$$\int_{\Omega} \frac{1}{\mu(w)} \mathrm{d}w < \infty$$

and

$$\int_{\Omega} \mu(w) \mathrm{d}w < \infty.$$

Let for any $z \in U$ and $c \in \mathbb{R}$ f denote minimal (z, c)-solution in weight μ of the Laplace's equation in $\mathbb{R}^n$. Then

$$|f(w)| \leq c\sqrt{\int_{\Omega} \mu(w) \mathrm{d}w}\sqrt{\int_{\Omega} \frac{1}{\mu(w)} \mathrm{d}w \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}\delta(w)^n}},$$

where $\delta(w)$ *denotes the distance of* w *to the boundary of* Ω*. In particular, if* μ *is equal to 1 almost everywhere, then*

$$|f(w)| \leq cL(\Omega)\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}\delta(w)^n},$$

where $L(\Omega)$ *denotes Lebesgue measure of* Ω*.*

Proof: First let us consider situation when $c = 1$. By Theorem 1.3

$$|f(w)| = \left|\frac{K(z,w)}{K(z,z)}\right|.$$

By Proposition 1.2

$$|f(w)| \leq \frac{\sqrt{K(w,w)}}{\sqrt{K(z,z)}}.$$

By Proposition 1.3 and (2.4)

$$\sqrt{K(w,w)} \leq \sqrt{\int_{B(z,r)} \frac{1}{\mu(w)} \mathrm{d}w \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}\delta(w)^n}}.$$

By Proposition 1.1

$$\frac{1}{\sqrt{K(z,z)}} \leq \sqrt{\int_\Omega \mu(w)\mathrm{d}w}.$$

If $c \neq 1$, $c \neq 0$, then a minimal (z,c)-solution in weight μ is equal to minimal $(z,1)$-solution in weight μ multiplied by c, as in the proof of theorems from the previous section.

If $c = 0$, then inequality from the theorem is trivial. ■

Chapter 3

Kernels of Szegö type

Kernels of Szegö type were introduced in [Szegö1921]. Their properties were widely examined, but not as wide the properties of the Bergman kernel.

Weighted Szegö kernel was investigated in few papers (see e.g. [Nehari1952], [Alenitsin1972], [Uehara1984], [Uehara1995]); the second one is in russian). In all of them, however, only continuous weights were in the consideration and we will not need this assumption in the dissertation.

As it was said in Chapter 1, case of the Szegö kernel is more subtle, so we will need to change general theory introduced there a bit to suit it to this case.

3.1 Poisson Kernel

Before we proceed, we need to recall the concept of a Poisson kernel, which will be used later. This section is mainly based on [Stein1972].

Definition 3.1. *Let Ω be a bounded domain with a boundary of class C^2 in $\mathbb{R}^n$. Function G defined on $\Omega \times \overline{\Omega} \setminus \{(x,y) \in \Omega \times \Omega | x = y\}$, such that*

(i) G is of class $C^{2-\varepsilon}$;

(ii) $\triangle_y G(x,y) = 0$ for $x \neq y$;

(iii) $G(x,y) - C_n|x-y|^{-n+2}$ *is harmonic in* $y \in \Omega$ *for each fixed* $x \in \Omega$*;*

(iv) $G(x,y)|_{y\in\partial\Omega} \equiv 0$*;*

is ***a Green's function*** *of a domain* Ω*.*

It can be proved that a function which satisfies all of these conditions is unique.

Definition 3.2. *Let* Ω *be a bounded domain with a boundary of class* C^2*. Function* $P : \Omega \times \partial\Omega \to \mathbb{R}$*, such that*

$$P(x,y) := -\frac{\partial G(x,y)}{\partial v_y}$$

for v_y *being an outward unit vector normal to* $\partial\Omega$ *at* y*, is* ***a Poisson kernel*** *of domain* Ω*.*

Theorem 3.1. *Let* $f : \overline{\Omega} \to \mathbb{R}$ *be harmonic on* Ω *and continuous on* $\overline{\Omega}$*. Then for any* $x \in \Omega$

$$f(x) = \int_{\partial\Omega} f(y)P(x,y)\mathrm{d}S,$$

where $\mathrm{d}S$ *denotes integral of a scalar field on* y*.*

We omit the proof.

Proposition 3.1. *A Poisson kernel for a ball in* $\mathbb{R}^n$ *with center* x *and radius* r *is given by*

$$P(z,w) = \frac{r^2 - |z-x|^2}{rC_r|z-w|^2},$$

where C_r *is the surface area of the unit (n-1)-sphere.*

We omit the proof.

3.2 Lax-Milgram Theorem

Here we recall classical results which will be used to prove that Szegö kernel depends in continuous way on a weight of integration.

In the whole section $\mathcal{H}$ will be an arbitrary Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle -, \cdot \rangle$ and a norm $|| \cdot ||$.

Theorem 3.2. *Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a sesquilinear form. The following conditions are equivalent:*

(i) B is continuous;

(ii) B is separately continuous on each variable;

(iii) there exists $C > 0$ such that

$$|B(x, y)| \leq C||x|| \cdot ||y||$$

for any $x, y \in \mathcal{H}$.

(iv) there exists $A \in B(\mathcal{H})$, such that

$$B(x, y) = \langle x|Ay \rangle$$

for any $x, y \in \mathcal{H}$.

We omit the proof.

Definition 3.3. *We will say that a sesquilinear form $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is* ***coercive****, if there exists $\alpha > 0$, such that*

$$B(x, x) \geq \alpha ||x||^2 \tag{3.1}$$

for any $x \in \mathcal{H}$.

Theorem 3.3. *(**Lax-Milgram**) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a sesquilinear form which is continuous and coercive on $\mathcal{H}$. Then for any $f \in \mathcal{H}$ there exists a unique element $g_f \in \mathcal{H}$, such that*

$$B(h, g_f) = \langle h, f \rangle$$

for all $h \in \mathcal{H}$. For such g_f we have

$$||g_f|| \leq \frac{1}{\alpha}||f||,$$

where $\alpha > 0$ is the smallest constant for which inequality (3.1) holds for any $h \in \mathcal{H}$.

We omit the proof.

Originally this result was published in [Lax1954]. Now we know that more general theorems are true (see e.g. [Babuška1971]), but we will not need them in the considerations.

3.3 Weighted Szegö space and weighted Szegö kernel

For $\mu : \partial\Omega \to \mathbb{R}$ measurable and almost everywhere greater than 0 (which we will call a **weight**) by $L^2(\partial\Omega, \mu)$ we will denote a set of classes of functions $f : \partial\Omega \to \mathbb{C}$, square-integrable in the sense

$$||f||_\mu^2 := \int_{\partial\Omega} |f(w)|^2 \mu(w) \mathrm{d}S < \infty,$$

where the integral is understood as an integral of a scalar field with a surface measure. The set $L^2(\partial\Omega, \mu)$ with an inner product given by

$$\langle f|g\rangle_\mu := \int_{\partial\Omega} \overline{f(w)} g(w) \mu(w) \mathrm{d}S$$

is a Hilbert space. Now let us consider the space $A(\Omega)$ of continuous functions $f : \overline{\Omega} \to \mathbb{C}$, such that $f_{|\Omega}$ is holomorphic. Let us denote $B(\Omega, \mu) := \{f_{|\partial\Omega} : f \in A(\Omega)\} \cap L^2(\partial\Omega, \mu)$. By $L^2H(\partial\Omega, \mu)$ we will understand the closure of $B(\Omega, \mu)$ in $L^2(\partial\Omega, \mu)$ topology.

Of course $L^2H(\partial\Omega, \mu)$ can change as a set with a change of μ. However,

Proposition 3.2. *If μ_1, μ_2 are weights and there exist $m, M > 0$, such that*

$$m\mu_1(z) \le \mu_2(z) \le M\mu_1(z) \tag{3.2}$$

a.e. on $\partial\Omega$, then for any $f \in L^2(\partial\Omega, \mu_j)$ we have $f \in L^2(\partial\Omega, \mu_k)$, $j, k \in \{1, 2\}$, and $m||f||_{\mu_1}^2 \le ||f||_{\mu_2}^2 \le M||f||_{\mu_1}^2$. Hence $L^2H(\partial\Omega, \mu_1) = L^2H(\partial\Omega, \mu_2)$ as a set and norms $||\cdot||_{\mu_1}$ and $||\cdot||_{\mu_2}$ are equivalent. In particular if $0 < m \le \mu \le M < \infty$, then $L^2H(\partial\Omega, \mu) = L^2H(\partial\Omega, 1)$ as a vector space.

Simple examples show that converse of these implications is not true.

If $L^2H(\partial\Omega, \mu_1) = L^2H(\partial\Omega, \mu_2)$ as vector spaces, we will write $\mu_1 \approx \mu_2$. It is easy to show that it is an equivalence relation.

Each element of $B(\Omega, 1)$ has a unique holomorphic prolongation to Ω by Theorem 3.1 (see [Krantz2002] for more details), so it is also true for any element from $B(\Omega, \mu)$, because $B(\Omega, \mu) \subset B(\Omega, 1)$ for any μ. We will denote the set of all such prolongations by $\tilde{B}(\Omega, \mu)$ (where $\tilde{B}(\Omega, \mu) \subset A(\Omega)$).

A good question to ask is how to find a holomorphic prolongation of functions from $L^2H(\partial\Omega, \mu) \setminus B(\Omega, \mu)$ for an arbitrary μ? We will answer this question in a moment.

We will use the same symbol for a function and its prolongation, which should not be misleading.

Let μ be a weight with the following property:

(CB) for any compact set $X \subset \Omega$ there exists $C_X > 0$, such that for any $f \in \tilde{B}(\Omega, \mu)$ and $z \in X$

$$|f(z)| \leq C_X ||f||_\mu.$$

Then for functions from $L^2H(\partial\Omega, \mu) \setminus B(\Omega, \mu)$ we can define their prolongation to Ω in the following way:

Let (f_n) be a sequence of functions from $\tilde{B}(\Omega, \mu)$. Let $f \in L^2H(\partial\Omega, \mu)$ be the limit of this sequence. Since by (CB) the sequence of functions $(f_{n|\Omega})$ fullfils the Cauchy condition locally uniformly on Ω, the function

$$f(z) := \lim_{n\to\infty} f_n(z), z \in \Omega$$

is well defined and holomorphic on Ω.

From now on, if μ fullfils (CB), we will interpret $L^2H(\partial\Omega, \mu)$ as a set of functions on $\overline{\Omega}$.

Definition 3.4. *Let μ be a weight satisfying (CB). A function (if it exists) $S_\mu : \Omega\times\overline{\Omega} \to \mathbb{C}$, such that for any $z \in \Omega$, $\overline{S_\mu(z,\cdot)} \in L^2H(\partial\Omega,\mu)$ and for any $f \in L^2H(\partial\Omega,\mu)$ (reproducing property)*

$$f(z) = \langle \overline{S_\mu(z,\cdot)} | f(\cdot)\rangle_\mu,$$

will be called ***Szegö kernel*** *of $L^2H(\partial\Omega,\mu)$.*

It is true (as for any reproducing kernel Hilbert space) that if S_μ and S'_μ are Szegö kernels of the same space, then $S_\mu = S'_\mu$ and if the Szegö kernel exists, then it is given uniquely by the formula

$$S_\mu(z,w) = \sum_{i\in I} \varphi_i(z)\overline{\varphi_i(w)},$$

where $\{\varphi_i\}_{i\in I}$ is an arbitrary complete orthonormal system of $L^2H(\partial\Omega,\mu)$. Hence for any $z, w \in \Omega$ we have $S_\mu(w,z) = \overline{S_\mu(z,w)}$ and by Hartogs Theorem on separate analyticity the function $\Omega \times \Omega' \ni (z,w) \mapsto S^0(z,w) := S_\mu(z,\overline{w})$ is holomorphic, where $\Omega' = \{w \in \mathbb{C}^N : \overline{w} \in \Omega\}$. So S_μ is real analytic on $\Omega\times\Omega$, holomorphic with respect to first N variables and antiholomorphic with respect to last N variables. Moreover for any $z \in \Omega$ we have $||\overline{S_\mu(z,\cdot)}||^2_\mu = ||S_\mu(\cdot,z)||^2_\mu = S_\mu(z,z)$. It is a natural question to ask, which conditions must μ satisfy in order to $L^2H(\partial\Omega,\mu)$ to be a reproducing kernel Hilbert space.

Definition 3.5. *We will say that a weight μ is* ***Szegö admissible*** *(S-admissible for short) if there exists Szegö kernel of $L^2H(\partial\Omega,\mu)$ space.*

Theorem 3.4. *μ is an S-admissible weight if and only if the condition (CB) is satisfied.*

Proof: $\Rightarrow$ comes directly from the definition 3.4. We can use Theorem 1.1 and Proposition 1.3 to show that in fact the smallest possible constant C_X in condition (CB) is

$$\max_{z\in X} \sqrt{S_\mu(z,z)}.$$

$\Leftarrow$ (CB) means that functionals of evaluation i. e. functionals

$$\tilde{E}_z : \tilde{B}(\Omega,\mu) \ni f \mapsto f(z) \in \mathbb{C}$$

are continuous. Since $B(\Omega,\mu)$ is dense in $L^2H(\partial\Omega,\mu)$ we can prolong $\tilde{E}_z$ to the functional $E_z \in L^2H(\partial\Omega,\mu)^*$ with the same majoring constant C_X for any $z \in \Omega$. By Riesz Representation Theorem for E_z it means that for $z \in \Omega$ there exists $\overline{e_z} \in L^2H(\partial\Omega,\mu)$, such that for any $f \in L^2H(\partial\Omega,\mu)$

$$f(z) = \langle \overline{e_z} | f \rangle$$

and the function

$$S_\mu(z,w) := e_z(w), (z,w) \in \Omega \times \overline{\Omega}$$

is the Szegö kernel of $L^2H(\partial\Omega,\mu)$. ■

Definition 3.6. *Classical Szegö space $H^2(\Omega)$ can be also defined as a set of holomorphic functions on Ω, for which*

$$\sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f(w)|^2 \mathrm{d}S < \infty,$$

where $\partial\Omega_\varepsilon = \{z \in \Omega : \delta(z) = \varepsilon\}$ and $\delta(z)$ denotes distance of z from $\partial\Omega$.

Existence of non-tangential limit $f(w)$ for almost every point $w \in \partial\Omega$ and any $f \in H^2(\Omega)$ is a classical result of theory of Hardy spaces (see e.g. [Stein1972] or [Stein1993]).

For any $f \in H^2(\Omega)$

$$\int_{\partial\Omega} |f(w)|^2 \mathrm{d}S = \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f(w)|^2 \mathrm{d}S < \infty \tag{3.3}$$

(see [Krantz2002] or [Stein1972] for more details).

Of course for weight $\mu : \partial\Omega \to \mathbb{R}$, such that $\mu > c > 0$ a.e. we have

$$L^2H(\partial\Omega,\mu) = \left\{ f \in H^2(\Omega) : \int_{\partial\Omega} |f(w)|^2 \mu(w) \mathrm{d}S < \infty \right\}.$$

3.4 Admissible weights

The content of this section is based mainly on [Żynda2020].

3.4.1 Sufficient conditions for a weight to be S-admissible

Theorem 3.5. *Let μ be a weight on $\partial\Omega$, such that*

$$\int_{\partial\Omega} \frac{1}{\mu} \mathrm{d}S < \infty.$$

Then μ satisfies condition (CB).

Before we proceed, we will need the following Lemma:

Lemma 3.1. *Let Ω_1, Ω_2 be bounded domains with C^2-smooth boundaries, such that $\Omega_1 \subset \Omega_2$ and $m \geq 1$. Then for any $f \in L^2H(\partial\Omega_2) = L^2H(\partial\Omega_2, 1)$ we have*

$$\int_{\partial\Omega_1} |f(w)|^m \mathrm{d}S \leq C \int_{\partial\Omega_2} |f(w)|^m \mathrm{d}S, \tag{3.4}$$

where

$$C = \left(\frac{2 \max\{P_2(z_0, w) | w \in \partial\Omega_2\}}{\min\{P_1(z_0, w) | w \in \partial\Omega_1\}} \right)^{\frac{1}{m}}$$

and P_1, P_2 is the Poisson kernel of Ω_1, Ω_2, respectively and z_0 is fixed point in Ω_1. In particular, C does not depend on $f \in L^2H(\partial\Omega_2)$.

It is a particular case of a Lemma 2.1 from article [Chen2011], which was proven for $m > 1$. It remains true, however, for $m = 1$, since authors of [Chen2011] follow the proof of Theorem 1 from [Stein1972] and in case of $m = 1$ we just need to change $f(y)\mathrm{d}\sigma(y)$ to a finite Borel measure on $\partial\Omega$.

Proof of the Theorem: Let $z_0 \in \Omega$ and let r be sufficiently small for $K_0 := K(z_0, 2r) := \{w \in \mathbb{C}^N : |z_0 - w| < 2r\}$ to lie with its boundary in Ω. Then by Mean Value Theorem for harmonic functions we have for $f \in \tilde{B}(\Omega, \mu)$, $z \in K(z_0, r)$ and $K := K(z, r)$

$$|f(z)| = C_1 \left| \int_{\partial K} f(w) \mathrm{d}S \right| \leq C_1 \int_{\partial K} |f(w)| \mathrm{d}S,$$

where $\frac{1}{C_1}$ is a measure of ∂K. By Lemma 2.4 we have

$$\int_{\partial K} |f(w)| \mathrm{d}S \leq C_0 \int_{\partial K_0} |f(w)| \mathrm{d}S \leq C_0 C_2 \int_{\partial\Omega} |f(w)| \mathrm{d}S,$$

where by Lemma 3.1 we can fix C_0 so it suits for any $K(z,r)$, for $z \in K(z_0,r)$. By Cauchy-Schwarz inequality,

$$\int_{\partial\Omega} |f(w)|\mathrm{d}S = \int_{\partial\Omega} |f(w)|\frac{\sqrt{\mu(w)}}{\sqrt{\mu(w)}}\mathrm{d}S \leq \sqrt{\int_{\partial\Omega} |f(w)|^2\mu(w)\mathrm{d}S}\sqrt{\int_{\partial\Omega} \frac{1}{\mu(w)}\mathrm{d}S}.$$

Finally,

$$|f(z)| \leq C_0C_1C_2\sqrt{\int_{\partial\Omega} \frac{1}{\mu(w)}\mathrm{d}S}\sqrt{\int_{\partial\Omega} |f(w)|^2\mu(w)\mathrm{d}S} \leq C_0C_1C_2C_3 \parallel f \parallel_{\mu} \leq C \parallel f \parallel_{\mu},$$

where C does not depend on $z \in K(z_0,r)$. Hence μ satisfies (CB).

Corollary 3.1. *If Ω is a bounded domain with a boundary of class C^2, then a weight μ defined on $\partial\Omega$ such that $\mu(z) \geq c > 0$ is an S-admissible weight.*

Theorem 3.6. *Let Ω be a bounded domain with the boundary of class C^2. Let μ_1, μ_2 be weights on $\partial\Omega$, such that μ_1 is S-admissible and $\mu_2 \geq \mu_1$ a.e. Then μ_2 is also S-admissible.*

Proof: If μ_1 is S-admissible, then for any compact set $X \subset \Omega$ there exists $C_X > 0$, such that for any $z \in X$ and any $f \in \tilde{B}(\Omega, \mu_1)$

$$|f(z)| \leq C_X ||f||_{\mu_1}.$$

Since

$$\int_{\partial\Omega} |f(w)|^2\mu_1(w)\mathrm{d}S \leq \int_{\partial\Omega} |f(w)|^2\mu_2(w)\mathrm{d}S,$$

we have that $\tilde{B}(\Omega, \mu_2) \subset \tilde{B}(\Omega, \mu_1)$ and that for any $f \in \tilde{B}(\Omega, \mu_2)$

$$|f(z)| \leq C_X ||f||_{\mu_2}. \blacksquare$$

In particular, if μ is an S-admissible weight, then also e^{μ} and μ^{μ} are admissible weights, because $e^x > x$ and $x^x > x$ almost everywhere on the interval $[0, +\infty[$.

Corollary 3.2. *Let Ω be a bounded domain with the boundary of class C^2. Let Ψ_1, Ψ_2 be weights on $\partial\Omega$ and let Ψ_1 be S-admissible. Then $\Psi_1 + \Psi_2$ is also an S-admissible weight. In particular sum of S-admissible weights on the same boundary is an S-admissible weight.*

Theorem 3.7. *Let Ω be a bounded domain with the boundary of class C^2. Let μ_1, μ_2 be S-admissible weights, such that $\mu_2 \geq C > 0$ a.e. Then $\mu_1 \cdot \mu_2$ is an S-admissible weight.*

Proof: If μ_1 is S-admissible, then for any compact set $X \subset \Omega$ there exists $C_X > 0$, such that for any $z \in X$ and any $f \in \tilde{B}(\Omega, \mu_1)$

$$|f(z)| \leq C_X ||f||_{\mu_1}.$$

Since

$$\int_{\partial\Omega} |f(w)|^2 \mu_1(w) \mathrm{d}S = \frac{1}{C} \int_{\partial\Omega} |f(w)|^2 \mu_1(w) C \mathrm{d}S \leq \frac{1}{C} \int_{\partial\Omega} |f(w)|^2 \mu_1(w) \mu_2(w) \mathrm{d}S,$$

we have that $\tilde{B}(\Omega, \mu_1\mu_2) \subset \tilde{B}(\Omega, \mu_1)$ and for any $f \in \tilde{B}(\Omega, \mu_1\mu_2)$

$$|f(z)| \leq C_X \frac{1}{\sqrt{C}} ||f||_{\mu_1\mu_2}. \blacksquare$$

Corollary 3.3. *Let μ be an S-admissible weight on the boundary $\partial\Omega$ of class C^2 of a bounded domain Ω and let $\alpha > 0$. Then $\alpha\mu$ is also an S-admissible weight.*

3.4.2 Non-admissible weight for the unit circle in $\mathbb{C}^1$

Z. Pasternak-Winiarski in [Pasternak1992] found an example of a weight which is not admissible for the case of the Bergman kernel. As we will see in a moment, a similiar construction allows us to find a weight which is not S-admissible.

In this section we are going to use Theorem 2.8 again.

Let us define $\Omega := K(0,1) = \{z \in \mathbb{C} : |z| < 1\}$,

$$A_n := \{z \in \mathbb{C} : |z| < 2^{-n}\} \cup \{z \in \mathbb{C} : |\mathrm{Im} z| < 2^{-n} \wedge 0 < \mathrm{Re} z < 1\}$$

and

$$M_n := (\overline{\Omega} \setminus A_n) \cup \overline{A_{n+1}}.$$

Moreover let $f_n : M_n \to \mathbb{C}$ be defined in the following way

$$f_n(w) := \begin{cases} 1 + \frac{1}{n} & \text{for} \quad w \in \overline{A_{n+1}} \\ 0 & \text{for} \quad w \in \overline{\Omega} \setminus A_n \end{cases}$$

By Theorem 2.8 there exist polynomials $g_n : \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$ such that $|f_n(w) - g_n(w)| < \frac{1}{n}$ for any $w \in M_n$. It implies that $|g_n(w)| < \frac{1}{n}$ for $w \in \overline{\Omega} \setminus A_n$ and $1 < |g_n(w)| < 1 + \frac{2}{n}$ for $w \in \overline{A_{n+1}}$. Now let us define polynomials

$$h_n(w) := \frac{g_n(w)}{g_n(0)}.$$

Since $|g_n(0)| > 1$, h_n is well defined, $\left(1 + \frac{2}{n}\right)^{-1} < |h_n(w)| < 1 + \frac{2}{n}$ on $\overline{A_{n+1}}$ and $|h_n(w)| < \frac{1}{n}$ on $\overline{\Omega} \setminus A_n$. Now let us denote $D_n := \partial\Omega \cap \overline{A_n}$. Then we may define a weight

$$\mu(w) := \begin{cases} 1 & \text{for} \quad w \in \partial\Omega \setminus D_1; \\ 0 & \text{for} \quad w = 1; \\ \min\left\{1, \frac{1}{|h_n(w)|^2}\right\} & \text{for} \quad w \in D_n \setminus D_{n+1} \end{cases} \tag{3.5}$$

Since μ is bounded from above (by 1), $h_n \in \tilde{B}(\Omega, \mu)$ for any $n \in \mathbb{N}$. It is not hard to show that for any $w \in \partial\Omega$

$$|h_n(w)|^2 \mu(w) < 9.$$

and

$$\lim_{n\to\infty} |h_n(w)|^2 \mu(w) = 0.$$

Therefore, by Lebesgue Majorized Convergence Theorem, we have:

$$\int_{\partial\Omega} \lim_{n\to\infty} |h_n(w)|^2 \mu(w) \mathrm{d}S = \lim_{n\to\infty} \int_{\partial\Omega} |h_n(w)|^2 \mu(w) \mathrm{d}S = 0.$$

As we can see, $|h_n(0)| = 1$ for any n, but $||h_n||_\mu \to 0$, so the functional of point evaluation $\tilde{E}_0 : \tilde{B}(\Omega, \mu) \ni f \mapsto f(0) \in \mathbb{C}$ is not continuous on $\tilde{B}(\Omega, \mu)$ and therefore μ is not an S-admissible weight.

3.4.3 Non-admissible weight for the unit ball in $\mathbb{C}^N$

Let $\Omega := K(0,1) = \{w \in \mathbb{C}^N : |w| < 1\}$ and $U := \{w \in \mathbb{C}^N : |w_1| \leq 1\}$. Let

$$A_n := \big(\{w \in \mathbb{C}^N : |w_1| < 2^{-n}\} \cup \{w \in \mathbb{C}^N : |\mathrm{Im} w_1| < 2^{-n} \wedge 0 < \mathrm{Re} w_1 < 1\}\big) \cap \overline{\Omega}$$

and

$$M_n := (\overline{\Omega} \setminus A_n) \cup \overline{A_{n+1}}.$$

Now we may define $p_n(w_1, w_2, \ldots, w_N) := h_n(w_1)$ on M_n, where $h_n : V \to \mathbb{C}$, $V := \{w \in \mathbb{C}^N : w_2 = \cdots = w_N = 0\}$, has the same properties and is constructed in the same way as in the previous section. Then we can define

$$\Psi(w_1, w_2, \ldots, w_N) := \begin{cases} 1 & \text{for} \quad w \in U \setminus A_1; \\ 0 & \text{for} \quad w_1 \in [0,1] \subset \mathbb{R}; \\ \min\left\{1, \frac{1}{|p_n(w)|^2}\right\} & \text{for} \quad w \in A_n \setminus A_{n+1}. \end{cases}$$

$\mu := \Psi_{|\partial\Omega}$ is non S-admissible weight on $\partial\Omega$. Indeed, since μ is bounded from above (by 1), $p_n \in \tilde{B}(\Omega, \mu)$ for any $n \in \mathbb{N}$ by Hartogs's Theorem on separate analytycity. Moreover for $w \in \partial\Omega$

$$|p_n(w)|^2 \mu(w) < 9.$$

and

$$\lim_{n \to \infty} |p_n(w)|^2 \mu(w) = 0.$$

Therefore, we can use Lebesgue Majorized Convergence Theorem as in the previous section, to show that functional of evaluation $\tilde{E}_0 : \tilde{B}(\Omega, \mu) \ni f \mapsto f(0) = f(0, 0, \ldots, 0) \in \mathbb{C}$ is not continuous on $\tilde{B}(\Omega, \mu)$. (Moreover all functionals of evaluation E_w for $w = (0, w_2, \ldots, w_N)$ are not continuous.)

3.4.4 Weights and biholomorphisms

In this section we are going to use the following theorems:

Theorem 3.8. *Let Ω_1, Ω_2 be open domains in $\mathbb{C}^N$ of one of the following types:*

Type 1: *smooth bounded pseudoconvex domain with the real analytic boundary;*

Type 2: *smooth bounded strictly pseudoconvex domain and (more generally);*

Type 3: *smooth bounded domain for which a $\overline{\partial}$-operator exists and satisfies subelliptic estimates.*

Then any biholomorphic mapping between Ω_1 and Ω_2 extends smoothly to the boundary.

This theorem was proved by S. Bell and E. Ligocka in [Bell1980]. Note that each (geometrically) convex domain is pseudoconvex and moreover in $\mathbb{C}^1$ each open domain is pseudoconvex. (See [Hörmander1990] or [Krantz2002] for more details.)

In the following two theorems we are going to use the same symbol for biholomorphism and its smooth prolongation to the boundary, if it exists, which should not be misleading.

Theorem 3.9. *Let Ω_1, Ω_2 be domains of one of types 1-3 introduced above and $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then for any integrable function $f : \partial\Omega_2 \to \mathbb{C}$ we have*

$$\int_{\partial\Omega_2} f \mathrm{d}S = \int_{\partial\Omega_1} (f \circ \Phi) |\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}} \mathrm{d}S,$$

where $J_{\mathbb{C}}\Phi$ is the complex Jacobian matrix of Φ.

It is a simple generalization of theorem included in [Barret2014] as Proposition 1. for particular f and for Ω_1, Ω_2 being strongly pseudoconvex domains with C^∞ boundary. It remains true in this version, since proof does not depend on integrated function and the reason for restriction to only strongly pseudoconvex domains was the fact that in that case biholomorphism has smooth prolongation to the boundary, which remains true in this more general case.

Theorem 3.10. *Let Ω_1, Ω_2 be of type 1, 2 or 3. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphism and μ be a weight on $\partial\Omega_2$. Then*

(i) for any g measurable and non-negative almost everywhere we have:

$$\int_{\partial\Omega_2} g\mu \mathrm{d}S < \infty \Leftrightarrow \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S < \infty$$

In particular, $h \in L^2H(\partial\Omega_2, \mu)$ if and only if $h \circ \Phi \in L^2H(\partial\Omega_1, \mu \circ \Phi)$.

(ii) μ is S-admissible on $\partial\Omega_2$ if and only if $\mu \circ \Phi$ is S-admissible on $\partial\Omega_1$.

Proof: *(i)* By the fact that $u := |\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}}$ is continuous function on compact set $\overline{\Omega_1}$, we have

$$C_1 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S \leq \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) |\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}} \mathrm{d}S$$
$$\leq C_2 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S,$$

where $C_1 := \min_{w \in \overline{\Omega}} u(w) > 0$ and $C_2 := \max_{w \in \overline{\Omega}} u(w)$. By Theorem 3.9 we can change integral in the middle to get:

$$C_1 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S \leq \int_{\partial\Omega_2} g\mu \mathrm{d}S \leq C_2 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S, \tag{3.6}$$

If the integral on the right hand side is finite, then integral in the middle must be also finite and if integral in the middle is finite, then integral on the left hand side must be also finite.

For the proof of the second part of *(i)* we just remind that a composition of two holomorphic functions is also a holomorphic function.

(ii) Since Φ is biholomorphism, we need only to show implication in one direction.

If μ is S-admissible on $\partial\Omega_2$, then for any compact set $X \subset \Omega_2$, $w \in X$ and any $f \in \tilde{B}(\partial\Omega_2, \mu)$ we have

$$|f(w)| \leq C_X \sqrt{\int_{\partial\Omega_2} |f|^2 \mu \mathrm{d}S}. \tag{3.7}$$

By using (3.6) for inequality (3.7) we gain

$$|(f \circ \Phi)(\tilde{w})| \leq C_X \sqrt{C_2} \sqrt{\int_{\partial\Omega_1} |f \circ \Phi|^2 (\mu \circ \Phi) \mathrm{d}S},$$

for $\Omega_1 \supset Y := \Phi^{-1}(X)$, $\tilde{w} := \Phi^{-1}(w) \in Y$, so (CB) is satisfied for $C_Y := C_X\sqrt{C_2}$. ■

Corollary 3.4. *For any simply-connected bounded domain Ω in $\mathbb{C}$ which is of type 1-3 there exists a non S-admissible weight on $\partial\Omega$.*

Proof: By Riemann Mapping Theorem there exists biholomorphism $\Phi : \Omega \to K(0,1)$. By Theorem 3.8 Φ has a smooth prolongation to $\partial\Omega$. By Theorem 3.10 the weight $\mu \circ \Phi$, where μ is a weight constructed in (3.5), is non S-admissible weight on $\partial\Omega$. ■

3.4.5 Weights on non-connected boundaries of domains

In this section we will prove theorem which states that in case of domain U in $\mathbb{C}^N$ such that ∂U is not connected "S-admissibility", in some sense, of a weight on one connected component of ∂U is sufficient for this weight to be S-admissible on whole ∂U.

Theorem 3.11. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with the boundary of class C^2. Let $G_1, \dots G_n$ be domains in $\mathbb{C}^N$ for $N \geq 2$, such that $\mathbb{C}^N \setminus G_j$ is connected, $\overline{G_j} \subset \Omega$, $\overline{G_j} \cap \overline{G_k} = \emptyset$ for $j \neq k$ and ∂G_j be of class C^2. Let μ be S-admissible weight on $\partial\Omega$ and let Ψ be a weight on ∂U, where $U := \Omega \setminus (\overline{G_1 \cup \dots \cup G_n})$, such that $\Psi(w) = \mu(w)$ for $w \in \partial\Omega$. Then Ψ is S-admissible weight on U. In addition, if $\Psi_{|G_j}$ is integrable on ∂G_j for any j, the map $L^2H(\partial\Omega, \mu) \ni f \mapsto Tf := f_{|U} \in L^2H(\partial U, \Psi)$ is a continuous isomorphism of Hilbert spaces.*

Proof: Let X be a compact subset in U. Then $X \subset \Omega$ and there exists $C_X > 0$, such that for any $f \in \tilde{B}(\Omega, \mu)$ and any $z \in X$

$$|f(z)| \leq C_X ||f||_\mu$$

On the other hand, if $g \in \tilde{B}(U, \Psi)$, then by Hartogs Prolongation Theorem, there exists $\tilde{g}$ continuous on $\overline{\Omega}$ and holomorphic on Ω, such that $\tilde{g}_{|U} = g$. It is obvious that

$$\int_{\partial\Omega} |\tilde{g}(w)|^2 \mu(w) \mathrm{d}S \leq \int_{\partial U} |\tilde{g}(w)|^2 \Psi(w) \mathrm{d}S = ||g||_\Psi^2 < \infty.$$

Then

$$||\tilde{g}||_\mu \leq ||g||_\Psi \tag{3.8}$$

and $\tilde{g} \in \tilde{B}(\Omega, \mu)$.

For any $z \in X$ we have

$$|g(z)| = |\tilde{g}(z)| \leq C_X||\tilde{g}||_\mu \leq C_X||g||_\Psi.$$

Since g is an arbitrary element of $\tilde{B}(U, \Psi)$, we see that Ψ is an S-admissible weight on ∂U.

Moreover, if $\Psi_{|\partial G_j}$ is integrable on any ∂G_j, then for any $f \in \tilde{B}(\Omega, \mu)$ we have that $f_{|\overline{U}} \in \tilde{B}(U, \Psi)$ and the prolongation $\tilde{B}(U, \Psi) \ni g \mapsto \tilde{g} \in \tilde{B}(\Omega, \mu)$ is unicly defined, then it is an inverse of T. By (3.8), T^{-1} is bounded and by Banach Inverse Theorem, T is continuous. Since condition (CB) is fulfilled, the same considerations can be applied to a function $f \in L^2H(\partial U, \Psi)$. ■

In the case $N = 1$, Theorem is not true. For example, if $\Omega := K := K(0,1) = \{w \in \mathbb{C} : |w| < 1\}$, $G := K(0, \frac{1}{2})$, $\mu \equiv 1$ and $\Psi \equiv 1$, the function

$$g(w) = \frac{1}{w},\ w \in U,$$

is an element of $L^2H(\partial U, \Psi)$, but it has no prolongation to a function $\tilde{g} \in L^2H(\partial\Omega, \mu)$. However, using similar argument as in the proof of the Theorem, we can show that if $N = 1$, then the operator of restriction $T : L^2H(\partial\Omega, \mu) \to L^2H(\partial U, \Psi)$ is continuous and one-to-one map onto its image, and that $T(L^2H(\partial\Omega, \mu))$ is a closed subspace of $L^2H(\partial U, \Psi)$.

3.5 Properties of the weighted Szegö kernel

Theorem 3.12. *If $f \in H(\overline{\Omega})$ is a function such that $f(z) \neq 0$ for any $z \in \overline{\Omega}$, then $\mu(z) := |f(z)|^2$ is an S-admissible weight on $\partial\Omega$, $L^2H(\partial\Omega, \mu) = L^2H(\partial\Omega, 1)$ as a set and the Szegö kernel S_μ of $L^2H(\partial\Omega, \mu)$ is equal to*

$$S_\mu(z, w) = \frac{1}{f(z)\overline{f(w)}} S_1(z, w),$$

where S_1 is the Szegö kernel of $L^2H(\partial\Omega, 1)$.

Proof: Because f is a continuous function on a compact set and $f(z) \neq 0$, $|f(z)| > c > 0$, on $\partial\Omega$, which means that $\mu(z) = |f(z)|^2$ is an S-admissible weight by corollary 3.1. Because f is a continuous function on a compact set, we also have $|f(z)| < C < \infty$, so $L^2H(\partial\Omega, 1)$ and $L^2H(\partial\Omega, \mu)$ are equal as sets in consequence of Proposition (3.2).

Since $\overline{S_1(z,\cdot)} \in L^2H(\partial\Omega, \mu)$ for any fixed $z \in \Omega$, it is also true that

$$\frac{1}{\overline{f(z)}}\frac{1}{f(\cdot)}\overline{S_1(z,\cdot)} \in L^2H(\partial\Omega, \mu),$$

as a product of two holomorphic functions, which is square-integrable on $\partial\Omega$, by definition 3.6.

Moreover, for any $g \in \tilde{B}(\Omega, \mu)$ we have

$$\begin{aligned}\langle\overline{S_\mu(z,\cdot)}|g\rangle_\mu &= \int_{\partial\Omega} g(w)\frac{1}{f(z)}\frac{1}{\overline{f(w)}}\overline{S_1(z,w)}|f(w)|^2 \mathrm{d}S(w)\\ &= \int_{\partial\Omega} g(w)\frac{1}{f(z)}f(w)\overline{S_1(z,w)}\mathrm{d}S(w).\end{aligned}$$

By the reproducing property of $S_1(z,\cdot)$ for functions from $L^2H(\partial\Omega, 1) = L^2H(\partial\Omega, \mu)$

$$\int_{\partial\Omega} g(w)\frac{1}{f(z)}f(w)\overline{S_1(z,w)}\mathrm{d}S(w) = g(z)\frac{f(z)}{f(z)} = g(z),$$

so $S_\mu(z,\cdot)$ has reproducing property for functions from $L^2H(\partial\Omega, \mu)$. ■

It is well-known that classical (i.e. for weight equal to 1) Szegö kernel for the unit ball in $\mathbb{C}^n$ is given by

$$S_1(z,w) = \frac{(n-1)!}{2\pi^n}\frac{1}{1-\langle z|\overline{w}\rangle^n}$$

See [Krantz2002] for more details. Using this fact and Theorem 3.12 allows us to give direct formula for the weighted Szegö kernel of the unit ball for weights which are square of modulus of holomorphic function.

Theorem 3.13. *Let μ_1, μ_2 be S-admissible weights on $\partial\Omega$, such that $\mu_1(w) \leq \mu_2(w)$ for almost every $w \in \partial\Omega$. Then*

$$S_{\mu_2}(z,z) \leq S_{\mu_1}(z,z)$$

for every $z \in \Omega$.

Proof: First let us assume that $S_{\mu_1}(z,z)$ and $S_{\mu_2}(z,z)$ are greater than 0. By Theorem 1.3 it is true that

$$\frac{1}{S_{\mu_1}(z,z)} = \int_{\partial\Omega} \left|\frac{S_{\mu_1}(z,w)}{S_{\mu_1}(z,z)}\right|^2 \mu_1(w)\mathrm{d}S \leq \int_{\partial\Omega} \left|\frac{S_{\mu_2}(z,w)}{S_{\mu_2}(z,z)}\right|^2 \mu_1(w)\mathrm{d}S.$$

Since $\mu_1 \leq \mu_2$,

$$\int_{\partial\Omega} \left|\frac{S_{\mu_2}(z,w)}{S_{\mu_2}(z,z)}\right|^2 \mu_1(w)\mathrm{d}S \leq \int_{\partial\Omega} \left|\frac{S_{\mu_2}(z,w)}{S_{\mu_2}(z,z)}\right|^2 \mu_2(w)\mathrm{d}S.$$

Because

$$\int_{\partial\Omega} \left|\frac{S_{\mu_2}(z,w)}{S_{\mu_2}(z,z)}\right|^2 \mu_2(w)\mathrm{d}S = \frac{1}{S_{\mu_2}(z,z)},$$

in conclusion we have that

$$\frac{1}{S_{\mu_1}(z,z)} \leq \frac{1}{S_{\mu_2}(z,z)},$$

which ends the proof.

Now let $S_{\mu_1}(z,z) = 0$. Then by Theorem 1.2 we have $S_{\mu_1}(z,\cdot) \equiv 0$. Since $\mu_1 \leq \mu_2$, we have $\overline{S_{\mu_2}(z,\cdot)} \in L^2H(U,\mu_1)$. Then by Theorem 1.2 again we have $S_{\mu_2}(z,\cdot) \equiv 0$, so $S_{\mu_2}(z,z) \leq S_{\mu_1}(z,z)$.

If $S_{\mu_2}(z,z) = 0$, then of course $S_{\mu_2}(z,z) \leq S_{\mu_1}(z,z)$. ■

3.6 Dependence of the Szegö kernel on a weight of integration

As it was stated before, the problem of dependence of weighted Szegö kernel on weights of integration was investigated in few articles. In all of them, however, only continuous weights were considered. Then it is natural to prove some theorems which state how weighted Szegö kernel depends on a weight in the case in which weight is not necessarily continuous.

At the beginning, we will introduce the appropriate topology in the set $\mathrm{SAW}(\partial\Omega)$ of

S-admissible weights on $\partial\Omega$. Let us recall that we denote by $\approx$ the equivalence relation on $\mathrm{SAW}(\partial\Omega)$ defined as follows: for any $\mu_1, \mu_2 \in \mathrm{SAW}(\partial\Omega)$ $\mu_1 \approx \mu_2$ if and only if $L^2H(\partial\Omega, \mu_1)$ is equal as a vector space to $L^2H(\partial\Omega, \mu_2)$. Just like at work [Maj2009] it can be proved that in this case norms in $L^2H(\partial\Omega, \mu_1)$ and $L^2H(\partial\Omega, \mu_2)$ are equivalent, i.e. there are positive constants c and C, such that for any $f \in L^2H(\partial\Omega, \mu_1)$ we have

$$c||f||_{\mu_2} \leq ||f||_{\mu_1} \leq C||f||_{\mu_2}.$$

For any $\mu \in \mathrm{SAW}(\partial\Omega)$ denote by $\mathrm{SAW}(\partial\Omega, \mu)$ the equivalence class of μ with respect to relation $\approx$. Note that $\mathrm{SAW}(\partial\Omega, \mu)$ contains infinitely many elements, because for any function $g \in L^\infty(\partial\Omega)$ such that

$$\mathrm{essinf}_{z\in\partial\Omega} g(z) > 0$$

the ordinary product $g\mu$ is an element of $\mathrm{SAW}(\partial\Omega, \mu)$ (see Proposition (3.2)) and if $g_1 \neq g_2$, then $g_1\mu \neq g_2\mu$.

On $\mathrm{SAW}(\partial\Omega, \mu)$ we consider the map:

$$\mathrm{SAW}(\partial\Omega, \mu) \ni \nu \mapsto B_\mu(\nu) := \langle -; \cdot \rangle_\nu \in \mathrm{Her}(L^2H(\partial\Omega, \mu)),$$

where $\mathrm{Her}(\mathcal{H})$ denotes the real Banach space of all continuous hermitian forms on a Hilbert space $\mathcal{H}$ with the standard Banach space norm:

$$||B|| := \sup_{||x||=||y||=1} |B(x, y)|, B \in \mathrm{Her}(\mathcal{H}).$$

We denote by τ_μ the weakest topology on $\mathrm{SAW}(\partial\Omega, \mu)$ with respect to which the map B_μ is continuous. By Lax-Milgram Theorem each inner (hermitian) product $\langle -|\cdot\rangle_\nu$ equivalent to $\langle -|\cdot\rangle_\mu$ on $L^2H(\partial\Omega, \mu)$ uniquely determines an invertible positive definite continuous operator A_ν on $L^2H(\partial\Omega, \mu)$, such that

$$\langle f|g\rangle_\nu = \langle A_\nu f|g\rangle_\mu, f, g \in L^2H(\partial\Omega, \mu).$$

Moreover, if $\mathrm{Her}_+(L^2H(\partial\Omega, \mu))$ denotes the cone in $\mathrm{Her}(L^2H(\partial\Omega, \mu))$ of all positive definite hermitian forms (the set of hermitian products) on $L^2H(\partial\Omega, \mu)$ equivalent to $\langle -|\cdot\rangle_\mu$, then the map

$$\mathrm{Her}_+(L^2H(\partial\Omega,\mu)) \ni \nu \mapsto \Psi_\mu(\nu) := A_\nu \in L(L^2H(\partial\Omega,\mu))$$

is an isometry (onto its image) with respect to standard norms in $\mathrm{Her}(L^2H(\partial\Omega,\mu))$ and in the space $L(L^2H(\partial\Omega,\mu))$ of all bounded endomorphisms of $L^2H(\partial\Omega,\mu)$. Therefore the map Ψ_μ is a homeomorphism. Hence τ_μ is the weakest topology with respect to which the map $\Psi_\mu \circ B_\mu$ is continuous.

On the other hand, if $\mu_1 \approx \mu$ and $\nu \in \mathrm{SAW}(\partial\Omega,\mu) = \mathrm{SAW}(\partial\Omega,\mu_1)$, then

$$\langle f|g\rangle_\nu = \langle (\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f|g\rangle_{\mu_1} = \langle (\Psi_\mu \circ B_\mu)(\mu_1)(\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f|g\rangle_\mu.$$

We can write

$$\langle (\Psi_\mu \circ B_\mu)(\mu_1)(\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f|g\rangle_\mu = \langle G \circ [(\Psi_{\mu_1} \circ B_{\mu_1})(\nu)] f|g\rangle_\mu$$

for $f, g \in L^2H(\partial\Omega,\mu)$, where $G : L(L^2H(\partial\Omega,\mu)) \to L(L^2H(\partial\Omega,\mu)$ is the map of composition with constant invertible operator

$$G(A) := [(\Psi_\mu \circ B_\mu)(\mu_1)] \circ A, A \in L(L^2H(\partial\Omega,\mu)).$$

Of course, such a map G is a homeomorphism of $L(L^2H(\partial\Omega,\mu))$. Hence $\Psi_\mu \circ B_\mu = G \circ (\Psi_{\mu_1} \circ B_{\mu_1})$ and therefore $\tau_\mu = \tau_{\mu_1}$. It means that the topology τ_μ does not depend on the choice of an equivalence class representative from $\mathrm{SAW}(\partial\Omega,\mu)$.

Let us consider the family $\bigcup_{\mu\in\mathrm{SAW}(\partial\Omega)} \tau_\mu$ of subsets of $\mathrm{SAW}(\partial\Omega)$. It is, of course, the base of same topology τ in $\mathrm{SAW}(\partial\Omega)$. From now on we will consider $\mathrm{SAW}(\partial\Omega)$ as a topological space endowed with this topology.

Note that for any $\mu \in \mathrm{SAW}(\partial\Omega)$ the set $\mathrm{SAW}(\partial\Omega,\mu)$ is open in $\mathrm{SAW}(\partial\Omega)$, but it is also closed, because

$$\mathrm{SAW}(\partial\Omega) \setminus \mathrm{SAW}(\partial\Omega,\mu) = \bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)\setminus\mathrm{SAW}(\partial\Omega,\mu)} \mathrm{SAW}(\partial\Omega,\nu).$$

Moreover from the definition of $\mathrm{SAW}(\partial\Omega)$ it follows almost immediately that for any $\nu_1, \nu_2 \in \mathrm{SAW}(\partial\Omega,\mu)$ and any $t \in [0,1]$ we have

$$t\nu_1 + (1-t)\nu_2 \in \mathrm{SAW}(\partial\Omega,\mu).$$

In addition, the map

$$[0,1] \ni t \mapsto (\Psi_\mu \circ B_\mu)(t\nu_1 + (1-t)\nu_2)) = t(\Psi_\mu \circ B_\mu)(\nu_1) + (1-t)(\Psi_\mu \circ B_\mu)(\nu_2) \in L(L^2H(\partial\Omega, \mu))$$

is evidently continuous and therefore the map

$$[0,1] \ni t \mapsto t\nu_1 + (1-t)\nu_2 \in \mathrm{SAW}(\partial\Omega, \mu)$$

is continuous. Hence $\mathrm{SAW}(\partial\Omega, \mu)$ is connected and consequently it is a connected component of $\mathrm{SAW}(\partial\Omega)$ with respect to τ.

It may happen for some $\mu \in \mathrm{SAW}(\partial\Omega)$ that B_μ is not $1-1$ map. In this case τ_μ is not a Haussdorf topology. In extreme cases it may happen that $L^2H(\partial\Omega, \mu) = \{0\}$ and $B_\mu \equiv 0$. On the other hand, in cases imporant for applications (for example, when μ is bounded from above and below by non-zero constants) B_μ is $1-1$ mapping and τ_μ is Haussdorf. Indeed, any weight bounded from above and below by non-zero constant is an element of $\mathrm{SAW}(\partial\Omega, 1)$, as a consequence of Proposition (3.2). For such μ all polynomials are elements of $L^2H(\partial\Omega, \mu)$ and it is easy to see that B_1 is an injection.

Our results are true in any case.

Now let us recall this theorem (see Theorem 5.1. in [Pasternak1998] for more details):

Theorem 3.14. *Let $\mathcal{H}$ be a Hilbert space and V be a closed vector subspace of $\mathcal{H}$. Let $P(\cdot)$ denote a mapping that assigns to each positive defined and invertible operator $A \in L(\mathcal{H})$ the projection of $\mathcal{H}$ onto V orthogonal with respect to the hermitian product*

$$\langle f|g\rangle_A := \langle f|Ag\rangle, f, g \in \mathcal{H}.$$

Then $P(\cdot)$ is analytic with respect to the natural analytic structure on an open set of all positively defined and invertible operators in $\mathcal{H}$.

We are ready to prove main results of this section.

Theorem 3.15. *For any $\mu \in \mathrm{SAW}(\partial\Omega)$ denote by S_μ the weighted Szegö kernel of $L^2H(\partial\Omega,\mu)$ defined on $\Omega \times \overline{\Omega}$. Then for any $\nu \in \mathrm{SAW}(\partial\Omega,\mu)$ and any $z \in \Omega$ the map*

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto \overline{S_\nu(z,\cdot)} \in L^2H(\partial\Omega,\mu)$$

is continuous with respect to the topology τ_μ on $\mathrm{SAW}(\partial\Omega,\mu)$ and the Hilbert space topology on $L^2H(\partial\Omega,\mu)$.

Proof: Fix $z \in \Omega$. We know that $\overline{S_\nu(z,\cdot)}$ is a vector representing functional of point evaluation $E_z : f \mapsto f(z)$ in the sense of Riesz Representation Theorem used for the space $L^2H(\partial\Omega,\mu)$ (i.e. in $L^2H(\partial\Omega,\mu)$ endowed with the inner product $\langle -|\cdot\rangle_\nu$). Let P_ν denote orthogonal projector onto $\ker E_z$ in $L^2H(\partial\Omega,\nu)$. It follows from the proof of Riesz Theorem that $\overline{S_\nu(z,\cdot)}$ can be expressed in terms of P_ν in this way:

Fix $f \in L^2H(\partial\Omega,\mu)$ (f does not depend on ν), such that $E_z = f(z) \neq 0$. (If $\ker E_z = L^2H(\partial\Omega,\mu)$, then $S_\nu(z,\cdot) = 0$ for any $\nu \in \mathrm{SAW}(\partial\Omega,\mu)$ by Theorem 1.2 and therefore our theorem is true.)

Let $g_\nu := (I - P_\nu)f$, where I denotes the identity operator of $L^2H(\partial\Omega,\mu)$. Then $g_\nu \neq 0$ for any $\nu \in \mathrm{SAW}(\partial\Omega,\mu)$. Using the fact that the subspace $(\ker E_z)^{\perp\nu}$ orthogonal to $\ker E_z$ in $L^2H(\partial\Omega,\nu)$ is one-dimensional, the following formula can be easily derived

$$\overline{S_\nu(z,\cdot)} = \frac{E_z(g_\nu)}{||g_\nu||_\nu^2} g_\nu = \frac{E_z((I-P_\nu)f)}{||(I-P_\nu)f||_\nu^2}(I-P_\nu)f.$$

In this formula E_z, I and f does not depend on ν. On the other hand, if $h \in L^2H(\partial\Omega,\mu)$, then by definition of τ_μ the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto ||h||_\nu^2 = B_\nu(h,h) \in \mathbb{R}$$

is continuous. (We used the same notation as in the definition of topology τ_ν.) Moreover, by standard arguments from the basic course of calculus, we get that if a map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto g_\nu \in L^2H(\partial\Omega,\mu)$$

is continuous, then the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto ||g_\nu||_\nu^2 = B_\nu(g_\nu,g_\nu) \in \mathbb{R}$$

is continuous.

To complete the proof of the theorem it is enough to show that the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto P_\nu \in L(L^2H(\partial\Omega,\mu))$$

is continuous. But this is a direct consequence of Theorem 3.14.

In our case, taking $\mathcal{H} = L^2H(\partial\Omega,\mu)$, $V = \ker E_z$ and $A = (\Psi_\mu \circ B_\mu)(\nu)$ for $\nu \in \mathrm{SAW}(\partial\Omega,\mu)$, we obtain that the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto P_\nu = P((\Psi_\mu \circ B_\mu)(\nu)) \in L(L^2H(\partial\Omega,\mu))$$

is continuous. This ends the proof of the theorem. ■

We could formulate Theorem 3.15 as follows: for any $z \in \Omega$ the map

$$\mathrm{SAW}(\partial\Omega) \ni \mu \mapsto \overline{S_\mu(z,\cdot)} \in \bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)} L^2H(\partial\Omega,\nu)$$

is continuous. This requires, however, introducing a topology on the set

$$\bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)} L^2H(\partial\Omega,\nu)$$

that is locally compatible with the topologies of Hilbert spaces on its components. It is possible, but we give up on it, because it complicates the considerations and is not needed in this thesis.

3.7 Applications to Complex Analysis

Content of this section is based mainly on [Żynda2019b].

Maximum modulus principle allows us to show that if two holomorphic functions are equal on a boundary of some bounded domain, then they are equal on the whole domain. In this section we will use the concept of weighted Szegö kernel to prove more general theorem:

Theorem 3.16. *Let Ω be a bounded domain with a boundary of finite measure and of class C^2. Let $f, g : \overline{\Omega} \to \mathbb{C}$, holomorphic on $\overline{\Omega}$ be functions such that $|f(z)| = |g(z)|$ on $\partial\Omega$ and $f(z), g(z) \neq 0$ for $z \in \overline{\Omega}$. Then $|f(z)| = |g(z)|$ for $z \in \overline{\Omega}$.*

Assumption that $f(z), g(z) \neq 0$ for $z \in \overline{\Omega}$ is necessary, because e.g. functions z^k and z^l for $k \neq l$ have the same modulus on $\partial K(0,1)$, but their modulus is not the same on whole $K(0,1) := \{z \in \mathbb{C} : |z| < 1\}$.

Proof of the Theorem: By Theorem 3.12 Szegö kernel of $L^2H(\partial\Omega, \mu)$ for $\mu = |f|^2$ is equal to

$$S_\mu(z,z) = \frac{1}{|f(z)|^2} S_1(z,z)$$

and at once

$$S_\mu(z,z) = \frac{1}{|g(z)|^2} S_1(z,z),$$

where S_1 is Szegö kernel of $L^2H(\partial\Omega, 1)$. So we have

$$\frac{1}{|f(z)|^2} S_1(z,z) = \frac{1}{|g(z)|^2} S_1(z,z).$$

$S_1(z,z)$ is given for any $z \in \Omega$, so that equality must be true on whole Ω. By Proposition 1.1, $S_1(z,z) > 0$, so we can divide both sides of the equation by $S_1(z,z)$ to get $|f(z)| = |g(z)|$ on whole Ω. ■

Corollary 3.5. *If $f : \overline{\Omega} \to \mathbb{C}$ is holomorphic on $\overline{\Omega}$ and $|f(z)| = c$ for $z \in \partial\Omega$ and $f(z) \neq 0$ for $z \in \Omega$, then f is constant on $\overline{\Omega}$.*

Proof: Function g equal to c on whole $\overline{\Omega}$ satisfies assumptions of the corollary. By Theorem 3.16, $|f(z)| = |g(z)|$ for any $z \in \Omega$, so $|f(z)| = c$ on whole Ω. By Riemann-Cauchy equations, if a holomorphic function f has constant modulus on some opein domain, then it is constant on the whole domain. ■

Note that Theorem 3.16 can be proved using elementary complex analysis. Indeed, if

f and g are holomorphic and non-zero on $\overline{\Omega}$, then $\frac{f}{g}$ and $\frac{g}{f}$ are also holomorphic and non-zero on $\overline{\Omega}$. We have $\left|\frac{f(z)}{g(z)}\right| = 1$ and $\left|\frac{g(z)}{f(z)}\right| = 1$ on $\partial\Omega$. By maximum modulus principle we conclude that $\left|\frac{f(z)}{g(z)}\right| = 1$ on $\overline{\Omega}$. ■

Note also that the unique function from Riemann Mapping Theorem can be described using classical (i.e. with a weight equal to 1 almost everywhere) Szegö kernel (see [Bell2015]).

3.8 Connection between Szegö and Poisson Kernels

Definition 3.7. *Let S_1 be a Szegö kernel of $L^2H(\partial\Omega, 1)$. Function $\mathcal{P}$ defined in the following way*

$$\mathcal{P}(z, w) := \frac{|S_1(z, w)|^2}{S_1(z, z)}$$

*is called **a Poisson-Szegö kernel** of domain Ω.*

It is easy to see that for any $f \in L^2H(\partial\Omega, 1)$ we have

$$\int_{\partial\Omega} \mathcal{P}(z, w) f(w) \mathrm{d}S(w) = f(z).$$

Theorem 3.17. *Let Ω be simply connected domain in $\mathbb{R}^2 = \mathbb{C}^1$ with a boundary of class C^2. Then*

$$P(z, w) = \mathcal{P}(z, w).$$

For the proof see [Bergman1970]. Note also that for higher dimensions $\mathcal{P}$ can never be expected to be equal to P, because their singularities are of different nature.

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